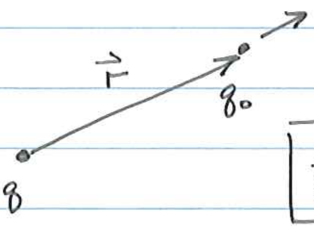
Electric field

$\vec{E}(\vec{r})$ is force per unit charge that a hypothetical test charge q_0 would feel at position $\vec{r} = (x, y, z)$

$$\vec{F}(\vec{r}) = q_0 \vec{E}(\vec{r}) \qquad \text{force acting on } q_0 \text{ at } \vec{r}.$$

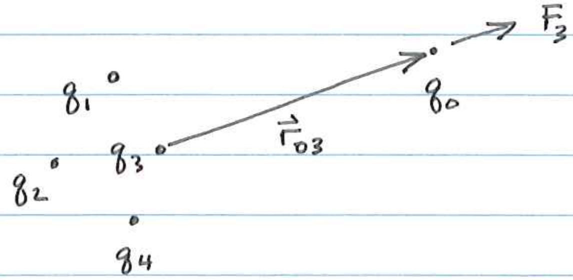
e.g. point charge q



$$\vec{F} = \frac{1}{4\pi\varepsilon_0} \frac{q q_0}{r^2} \hat{r} = q_0 \vec{E}$$

$$\boxed{\vec{E} = \frac{1}{4\pi\varepsilon_0} \frac{q}{r^2} \hat{r}}$$

e.g. collection of N point charges



total force $\vec{F} = \sum_{i=1}^{N} \vec{F}_i = \sum_{i=1}^{N} (\text{force on } q_0 \text{ from } q_i)$

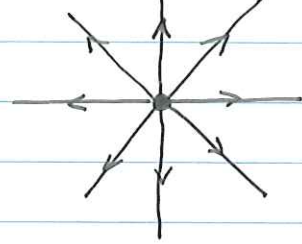
$$= \sum_{i=1}^{N} \frac{1}{4\pi\varepsilon_0} \frac{q_0 q_i}{r_{0i}^2} \hat{r}_{0i}$$

total electric field $= \vec{E} = \sum_{i=1}^{N} \frac{1}{4\pi\varepsilon_0} \frac{q_i}{r_{0i}^2} \hat{r}_{0i}$

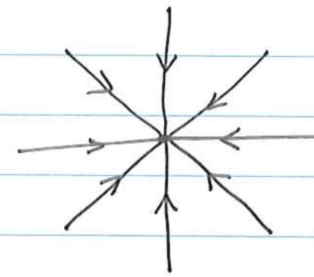
Note: $\vec{E}$ exists at every point, even when there is no test charge (it was hypothetical)

Electric field lines: visual representation of $\vec{E}$.

point charge:

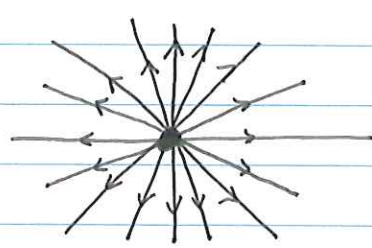


$q > 0$
(positive charge)



$q < 0$
(negative charge)

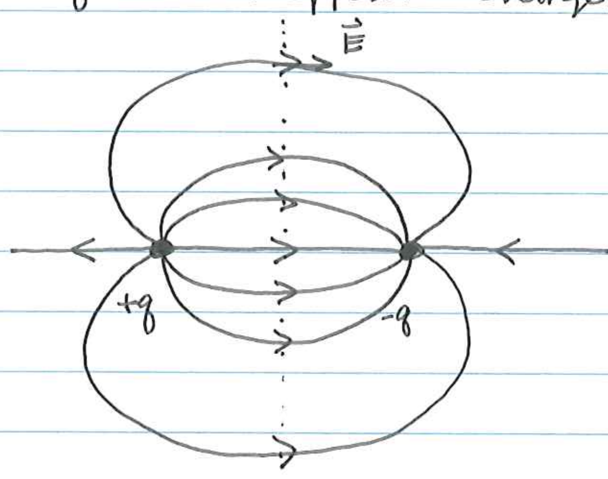
- lines show direction of $\vec{E}$ at each point.
- must be smooth & continuous; must only begin at positive charge, end at negative charge.
- number of lines per unit volume represents magnitude of $\vec{E}$.



$2q$

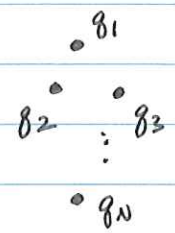
twice as many lines for twice the charge.

electric dipole (equal-and-opposite charges)



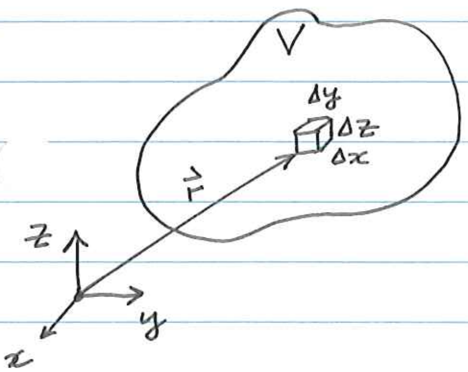
Continuous charge distribution

Instead of discrete charges $q_1, q_2, \ldots, q_N$, what if we have a continuous density of charge?



Total charge $Q = \sum_{i=1}^{N} q_i$

Consider a volume region V with a distribution of charge



$\rho(\vec{r}) = \dfrac{\text{charge}}{\text{volume}}$ = charge density at $\vec{r}$

Divide up V into tiny cubes with ~~volume~~ dimensions $\Delta x \times \Delta y \times \Delta z$ and volume $\Delta x \Delta y \Delta z$

The charge inside is $\Delta Q = \rho(\vec{r})\, \Delta x \Delta y \Delta z$.

The total charge is summed over all cubes.

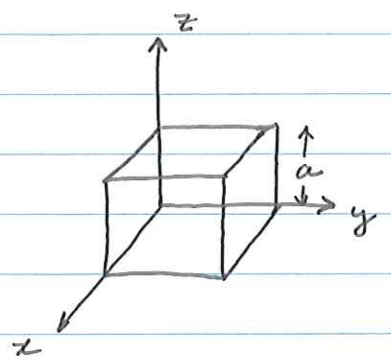
$$Q = \sum \Delta Q = \sum \rho(\vec{r})\, \Delta x\, \Delta y\, \Delta z$$

Limit where cubes become infinitesimally small: $\sum \to \int$

$\Delta x \to dx$, $\Delta y \to dy$, $\Delta z \to dz$.

$$Q = \int dx \int dy \int dz\; \rho(\vec{r}) = \int \underbrace{d^3r}_{\text{Volume integral.} = \int dV} \rho(\vec{r})$$

e.g. cube of length a with uniform charge density $\rho(\vec{r}) = \rho_0$



$$Q = \int d^3r \, \rho(\vec{r}) = \int_0^a dx \int_0^a dy \int_0^a dz \, \rho_0 = \rho_0 a^3$$

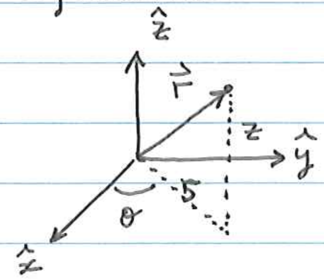
Same cube but with $\rho(\vec{r}) = \rho_0 x/a$

$$Q = \int_0^a dx \int_0^a dy \int_0^a dz \, \rho_0 x/a = \rho_0 a \int_0^a dx \, x = \frac{1}{2} \rho_0 a^3$$

Sometimes integrals are more convenient in different coordinate systems:

1) Cartesian: $\vec{r} = (x, y, z)$ $\qquad \int d^3r = \int dx \int dy \int dz$

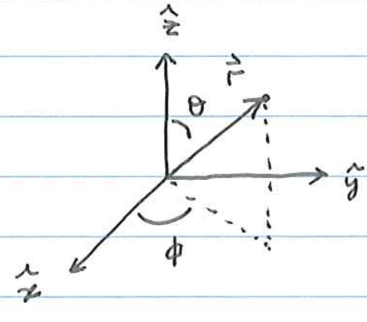
2) Cylindrical: (ς, θ, z) $\qquad \int d^3r = \int \varsigma \, d\varsigma \int_0^{2\pi} d\theta \int dz$



$$\varsigma = \sqrt{x^2 + y^2}$$

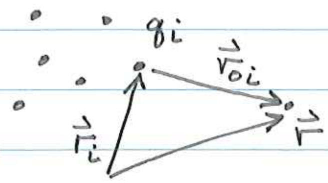
3) Spherical: (r, θ, ϕ)

$$\int d^3r = \int r^2 dr \int_0^\pi \sin\theta \, d\theta \int_0^{2\pi} d\phi$$

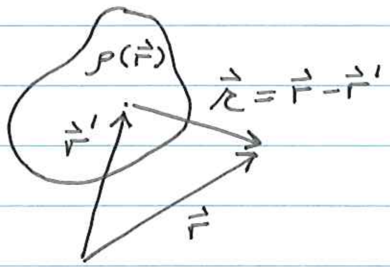


$$r = \sqrt{x^2 + y^2 + z^2}$$

Electric field:



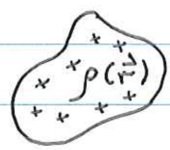
$$\vec{E}(\vec{r}) = \sum_{i=1}^{N} \frac{1}{4\pi\varepsilon_0} \frac{q_i}{r_{0i}^2} \hat{r}_{0i}$$



$$\vec{E}(\vec{r}) = \int d^3r' \, \frac{1}{4\pi\varepsilon_0} \frac{\rho(\vec{r}\,')}{\mathcal{R}^2} \hat{\mathcal{R}}$$

We can also have other types of continuous charge distributions:

volume charge (3D)

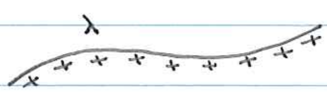


$$\rho = \frac{\text{charge}}{\text{volume}} = \text{charge density}$$

surface charge (2D)

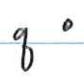
$$\sigma = \frac{\text{charge}}{\text{area}} = \text{surface charge density}$$

line charge (1D)



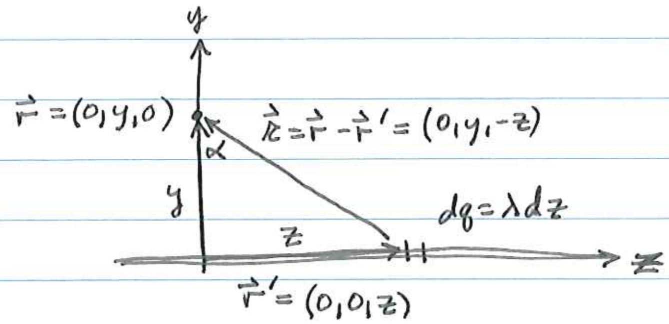
$$\lambda = \frac{\text{charge}}{\text{length}} = \text{linear charge density}$$

point charge (0D)



example: electric field of infinite line charge along z-axis

assume $\lambda = \text{const.}$



$$\mathscr{r} = \sqrt{y^2 + z^2}$$

$$\hat{\mathscr{r}} = \frac{1}{\sqrt{y^2+z^2}} \begin{pmatrix} 0 \\ y \\ -z \end{pmatrix}$$

$$d\vec{E} = \frac{1}{4\pi\varepsilon_0} \frac{dq}{\mathscr{r}^2} \hat{\mathscr{r}} = \frac{1}{4\pi\varepsilon_0} \frac{\lambda dz}{(y^2+z^2)^{3/2}} \begin{pmatrix} 0 \\ y \\ -z \end{pmatrix}$$

$$\vec{E} = \int d\vec{E} = \int_{-\infty}^{\infty} dz \frac{\lambda}{4\pi\varepsilon_0} \frac{1}{\mathscr{r}^2} \hat{\mathscr{r}}$$

only y-component contributes by symmetry.

$$E_y = \int_{-\infty}^{\infty} dz \frac{\lambda}{4\pi\varepsilon_0} \frac{y}{(y^2+z^2)^{3/2}}$$

Do integral using trig sub: $z = y \tan\alpha$

$dz = y \sec^2\alpha \, d\alpha$

$z = \infty \rightarrow \alpha = \pi/2$
$z = -\infty \rightarrow \alpha = -\pi/2$ } limits of integration

$$E_y = \frac{\lambda}{4\pi\varepsilon_0} \int_{-\pi/2}^{\pi/2} y \sec^2\alpha \, d\alpha \; y \frac{1}{(y^2 + y^2\tan^2\alpha)^{3/2}}$$

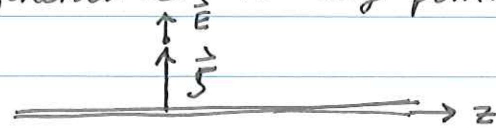
$$= \frac{\lambda}{4\pi\varepsilon_0 y} \int_{-\pi/2}^{\pi/2} d\alpha \frac{\sec^2\alpha}{(1+\tan^2\alpha)^{3/2}} = \frac{\lambda}{4\pi\varepsilon_0 y} \int_{-\pi/2}^{\pi/2} d\alpha \cos\alpha$$

using $1 + \tan^2\alpha = \sec^2\alpha$

$$= \frac{\lambda}{2\pi\varepsilon_0 y}$$

We have $\vec{E} = \dfrac{\lambda}{2\pi\varepsilon_0 y}\hat{y}$ at $\vec{r} = (0, y, 0)$

Can generalize to any point using cylindrical coords (s, θ, z)

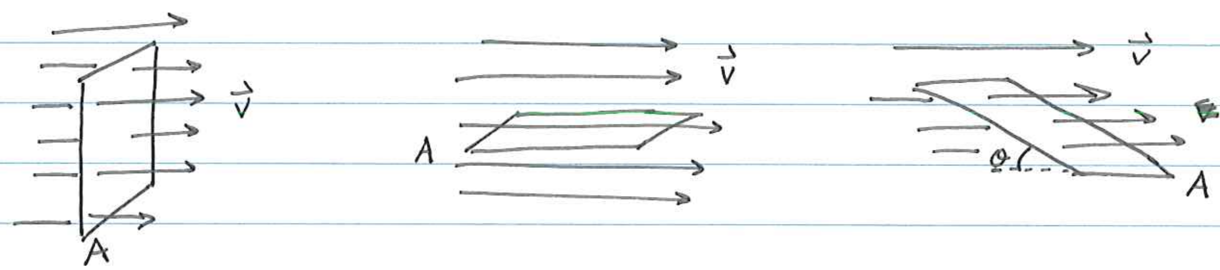


$$\vec{E} = \frac{\lambda}{2\pi\varepsilon_0 s}\hat{s}$$

Electric Flux

Flux Φ is the rate of flow of something through a given area A.

Consider a uniform stream of fluid ~~through~~ with velocity $\vec{v}$.



$\vec{v}$ perp. to A

$\Phi = |\vec{v}|A$

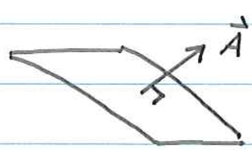
$\vec{v}$ parallel to A

$\Phi = 0$

$\Phi = |\vec{v}|A\cos\theta$

Only component of flow through A contributes.

Define an area vector $\vec{A}$ such that $A = |\vec{A}|$ and direction of A points normal to the surface.



Then $\Phi = \vec{v} \cdot \vec{A} = |\vec{v}|A\cos\theta$