We have $\vec{E} = \frac{\lambda}{2\pi\varepsilon_0 y}\hat{y}$ at $\vec{r} = (0, y, 0)$

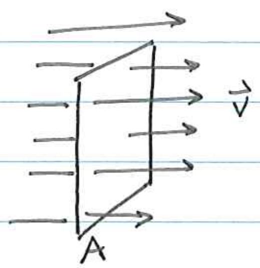
Can generalize to any point using cylindrical coords (s, θ, z)

$$\vec{E} = \frac{\lambda}{2\pi\varepsilon_0 s}\hat{s}$$

Electric Flux

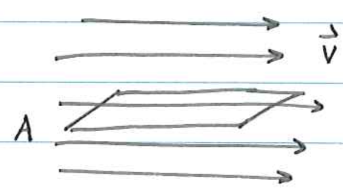
Flux Φ is the rate of flow of something through a given area A.

Consider a uniform stream of fluid ~~through~~ with velocity $\vec{v}$.



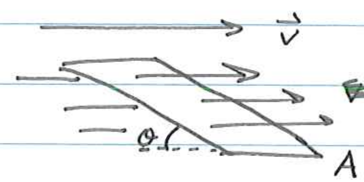
$\vec{v}$ perp. to A

$\Phi = |\vec{v}|A$



$\vec{v}$ parallel to A

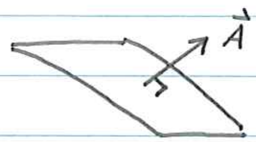
$\Phi = 0$



$\Phi = |\vec{v}|A\cos\theta$

only component of flow through A contributes.

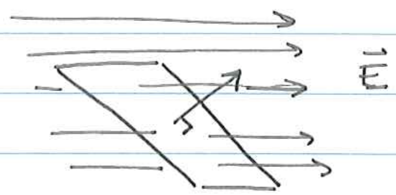
Define an area vector $\vec{A}$ such that $A = |\vec{A}|$ and direction of A points normal to the surface.



Then $\Phi = \vec{v}\cdot\vec{A} = |\vec{v}|A\cos\theta$

Electric flux Φ_E = Electric field "flowing" through a surface

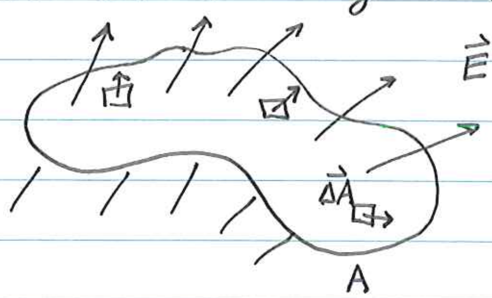
Consider a uniform $\vec{E}$ and a flat surface $\vec{A}$.



$$\Phi_E = \vec{E} \cdot \vec{A}$$

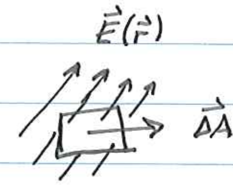
Need to generalize Φ_E to case where $\vec{E}$ is not uniform and A is not flat:

Consider an arbitrary surface A with an arbitrary $\vec{E}$:



Can divide A into small patches $\vec{\Delta A}$. Each $\vec{\Delta A}$ points normal to surface at that point $\vec{r}$.

Assume $\vec{\Delta A}$ is small enough such that $\vec{E}$ is uniform. Then the flux through each ΔA is:



$$\Delta \Phi_E = \vec{E}(\vec{r}) \cdot \vec{\Delta A}$$

The total flux is the sum over all patches $\vec{\Delta A}$:

$$\Phi_E = \sum \Delta \Phi_E = \sum \vec{E}(\vec{r}) \cdot \vec{\Delta A}$$

Infinitesimal limit: $\vec{\Delta A} \longrightarrow d\vec{A}$, $\quad \sum \longrightarrow \int$ (integral)

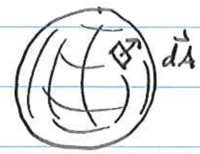
$$\Phi_E = \int_A d\vec{A} \cdot \vec{E}(\vec{r})$$

This is called a <u>surface integral</u>.

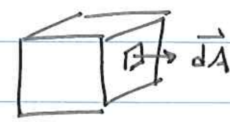
The surface A can be <u>open</u> or <u>closed</u>.

Closed surface: A has no boundary

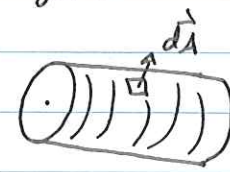
e.g. sphere



cube



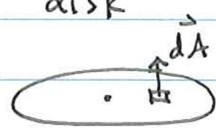
cylinder



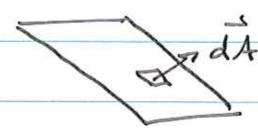
convention: $\vec{dA}$ points <u>outward</u>

open surface: A has a boundary

e.g. disk

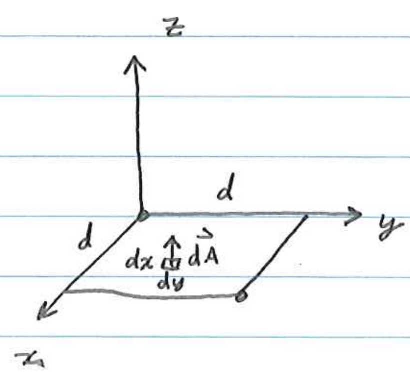


flat rectangle



can choose which way $\vec{dA}$ points (up or down)

example: let A be a square of side length d in the x-y plane as follows:



$\vec{dA} = dx\, dy\, \hat{z}$ points in z-direction

First let's compute $\vec{A}$: Should get $\vec{A} = d^2\hat{z}$

$$\vec{A} = \int d\vec{A} = \int dx\,dy\,\hat{z} = \int_0^d dx \int_0^d dy\;\hat{z} = d^2\,\hat{z} = A\,\hat{z}$$

Next, introduce $\vec{E}$ field and compute Φ_E:

$$\Phi_E = \int_A d\vec{A}\cdot\vec{E} = \int_0^d dx \int_0^d dy\; \underbrace{E_z(x,y,z=0)}_{\vec{E}\cdot\hat{z} = E_z}$$

example: $E_z(\vec{r}) = a\,|\vec{r}|^2$ where a is a constant.

$$= a\,(x^2+y^2+z^2)$$

$$\Phi_E = \int_0^d dx \int_0^d dy\; a\,(x^2+y^2)$$

$$= \int_0^d dx \left\{ a\,(x^2 y + \tfrac{1}{3}y^3)\Big|_{y=0}^{y=d} \right\}$$

$$= \int_0^d dx\; a\,(x^2 d + \tfrac{1}{3}d^3) = a\,(\tfrac{1}{3}d\,x^3 + \tfrac{1}{3}d^3 x)\Big|_0^d$$

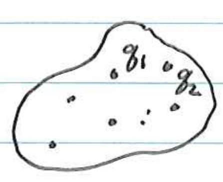
$$= \frac{2a}{3}\,d^4$$

Gauss's Law

Consider any closed surface A. The electric flux through A is

$$\boxed{\Phi_E = Q_{encl}/\varepsilon_0}$$

where Q_{encl} is the total electric charge enclosed within A.



A

$$Q_{encl} = \sum_{i=1}^{N} q_i \quad \text{for point charges}$$

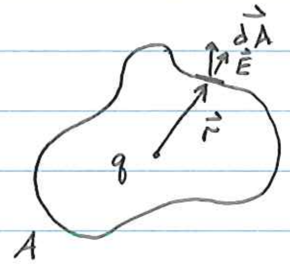
$$= \int_V d^3r\,\rho(\vec{r}) \quad \text{for continuous charge}$$

where V is the volume within the surface A.

Proof of Gauss's law

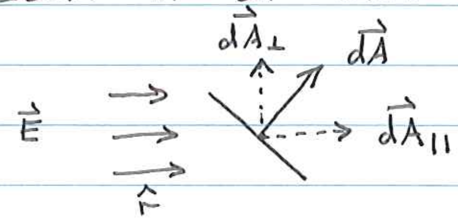
First prove it for a point charge, then generalize to any charge distribution using superposition principle.

Consider a point charge q enclosed within an arbitrary closed surface A.



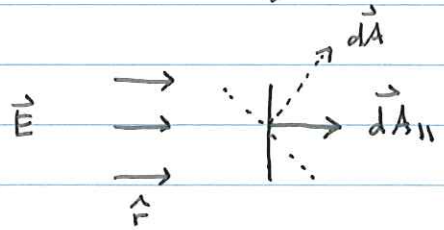
$$\vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\hat{r}$$

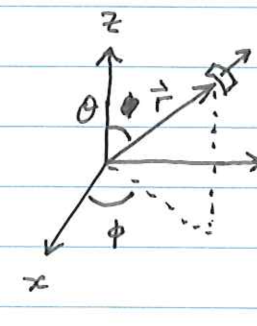
Zoom in on $\vec{dA}$:



divide $\vec{dA}$ into $\vec{dA} = \vec{dA}_\perp + \vec{dA}_\parallel$ where $\vec{dA}_\parallel$ is parallel to $\hat{r}$ and $\vec{dA}_\perp$ is perp. to $\hat{r}$.
Only $\vec{dA}_\parallel$ contributes to Φ_E.

⇓ same as





$\vec{dA}_\parallel = r^2 \sin\theta \, d\theta \, d\phi \, \hat{r}$ in spherical coords

Contribution to Φ_E from $\vec{dA}$ is

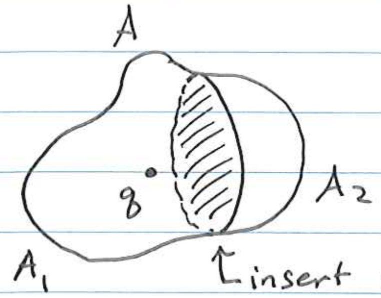
$$d\Phi_E = \vec{E}(\vec{r})\cdot\vec{dA} = \vec{E}(\vec{r})\cdot\vec{dA}_\parallel = \frac{q}{4\pi\varepsilon_0}\frac{1}{r^2} r^2 \sin\theta \, d\theta \, d\phi$$

$$= \frac{q}{4\pi\varepsilon_0}\sin\theta \, d\theta \, d\phi \qquad \text{(r - dependence cancels out!)}$$

Now compute Φ_E by integrating over θ, ϕ:

$$\Phi_E = \int d\Phi_E = \int_0^{\pi} d\theta \sin\theta \int_0^{2\pi} d\phi \frac{q}{4\pi\varepsilon_0} = \frac{q}{4\pi\varepsilon_0} \times 4\pi = \frac{q}{\varepsilon_0}$$

This is for <u>any</u> closed surface A enclosing q.
What if a closed surface doesn't enclose q?



↑ insert a "wall" to divide A into two smaller closed surfaces A_1 & A_2.

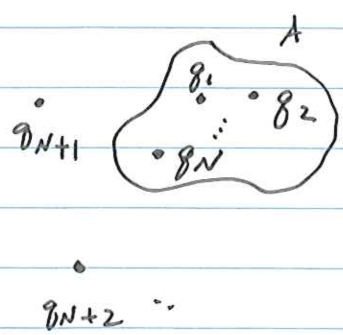
We have $\Phi_E(A) = \Phi_E(A_1) + \Phi_E(A_2)$
Flux through A = flux through A_1 + flux through A_2

Since $\Phi_E(A_1) = \frac{q}{\varepsilon_0}$ and $\Phi_E(A) = \frac{q}{\varepsilon_0}$, we must have $\Phi_E(A_2) = 0$.

~~$\Phi_E = 0$ if zero not charge is enclosed.~~

So $\Phi_E = 0$ if zero charge is enclosed.

Generalize to arbitrary charge distribution:



Since $\vec{E} = \vec{E}_1 + \vec{E}_2 + \ldots + \vec{E}_N + \vec{E}_{N+1} + \ldots$

$$\Phi_E = \int_A d\vec{A} \cdot \vec{E} = \int_A d\vec{A} \cdot (\vec{E}_1 + \vec{E}_2 + \ldots)$$

$$= \frac{q_1}{\varepsilon_0} + \frac{q_2}{\varepsilon_0} + \ldots + \frac{q_N}{\varepsilon_0} = \frac{Q_{encl}}{\varepsilon_0}$$

only enclosed charges contribute to Φ_E.

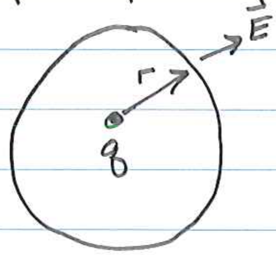
Applications of Gauss's law

Used to derive $\vec{E}$ for certain charge configurations.

Find a surface where $\vec{E}$ is constant & normal to the surface $\Longrightarrow$

$$\Phi_E = \int d\vec{A}\cdot\vec{E} = E\,A = \frac{Q_{encl}}{\varepsilon_0} \Rightarrow E = \frac{Q_{encl}}{\varepsilon_0 A}$$

example: point charge q



Draw Gaussian surface of radius r. By spherical symmetry, $\vec{E}$ must point in $\hat{r}$ direction. Always normal to the surface. Also $E = \text{const}$ over surface

$$\Phi_E = \int d\vec{A}\cdot\vec{E} = \int dA\,E = E\int dA = E\cdot \underbrace{4\pi r^2}_{\text{surface area}} = \frac{q}{\varepsilon_0}$$

So $\underline{E(r) = \dfrac{q}{4\pi\varepsilon_0 r^2}}$. Put in direction $\underline{\vec{E}(\vec{r}) = \dfrac{q}{4\pi\varepsilon_0 r^2}\hat{r}}$

example: spherical shell with uniform surface charge σ and radius R. Find $\vec{E}(\vec{r})$.

Total charge $Q = \sigma\cdot 4\pi R^2$

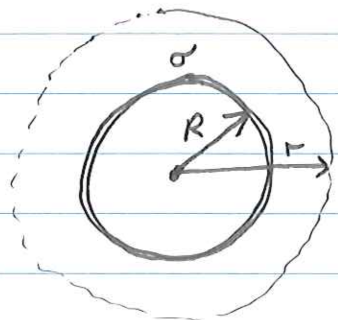
Consider two cases: $r < R$ and $r > R$.

$r < R$ (inside shell):

$$\Phi_E = EA = E\,4\pi r^2 = \frac{Q_{encl}}{\varepsilon_0} = 0$$

$$\Rightarrow \quad E = 0 \quad \text{or} \quad \vec{E} = 0.$$

$r > R$ (outside shell):



$$\Phi_E = EA = E\,4\pi r^2 = \frac{Q}{\varepsilon_0}$$

$$\Rightarrow \quad \underline{E = \frac{Q}{4\pi\varepsilon_0 r^2}} \quad \text{or} \quad \underline{\vec{E} = \frac{Q}{4\pi\varepsilon_0 r^2}\hat{r}}$$

Final answer:

$$\vec{E}(\vec{r}) = \begin{cases} 0 & r < R \\ \dfrac{\sigma R^2}{\varepsilon_0 r^2}\hat{r} & r > R \end{cases}$$

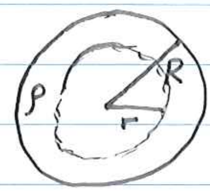
$$\frac{\sigma R^2}{\varepsilon_0 r^2}\hat{r} = \frac{Q}{4\pi\varepsilon_0 r^2}\hat{r}$$

outside R, $\vec{E}$ same as point charge

example: uniform charged sphere of charge density ρ and radius R.

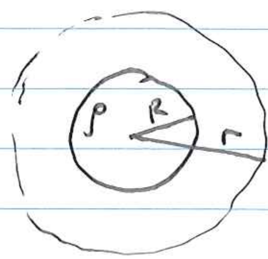
Total charge: $Q = \frac{4}{3}\pi R^3 \rho$

$r < R$ (inside sphere):



$$\Phi_E = EA = E\, 4\pi r^2 = \frac{Q_{encl}}{\varepsilon_0}$$
$$= \frac{1}{\varepsilon_0}\, \rho\, \frac{4\pi}{3} r^3$$
$$\Rightarrow \quad E = \frac{1}{4\pi\varepsilon_0 r^2}\, \rho\, \frac{4\pi}{3} r^3$$

$r > R$ (outside sphere):



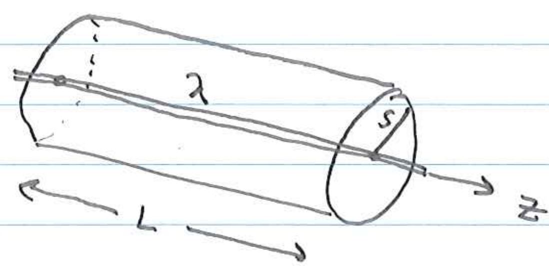
$$\Phi_E = E \cdot 4\pi r^2 = \frac{Q_{encl}}{\varepsilon_0} = \frac{1}{\varepsilon_0}\, \rho\, \frac{4\pi}{3} R^3$$
$$\Rightarrow \quad E = \frac{Q}{4\pi\varepsilon_0 r^2}$$

Same as point charge outside sphere.

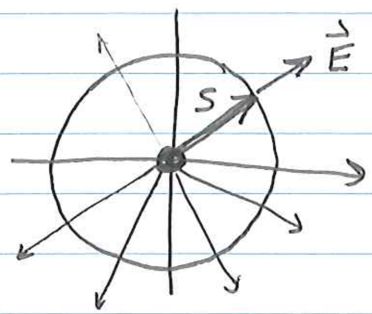
So $$E = \begin{cases} \dfrac{\rho r}{3\varepsilon_0} & r < R \\[2ex] \dfrac{\rho R^3}{3\varepsilon_0 r^2} & r > R \end{cases}$$

and: $$\vec{E} = E\hat{r} = \begin{cases} \dfrac{\rho r}{3\varepsilon_0}\, \hat{r} & r < R \\[2ex] \dfrac{\rho R^3}{3\varepsilon_0 r^2}\, \hat{r} & r > R \end{cases}$$

example: infinite wire of uniform linear charge density λ



Use cylindrical coords to draw Gaussian surface as a cylinder of radius s and length L.

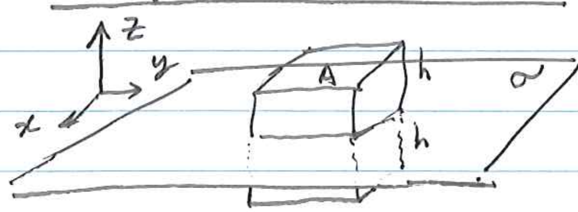


By symmetry, $\vec{E}$ must point in $\hat{s}$ direction, which is normal to the cylinder (except at ends)

$$\Phi_E = \int \vec{E}\cdot d\vec{A} = E(s)\underbrace{2\pi s L}_{\substack{\text{surface area}\\ \text{where } \vec{E} \text{ is } \perp}} = \frac{Q_{encl}}{\varepsilon_0} = \frac{\lambda L}{\varepsilon_0}$$

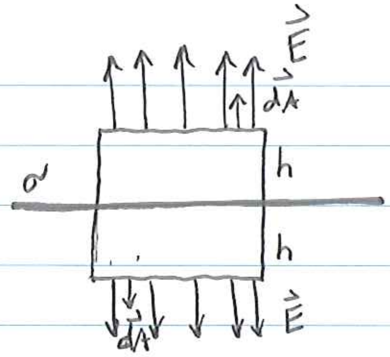
So $\underline{E(s) = \dfrac{\lambda}{2\pi \varepsilon_0 s}}$ $\underline{\vec{E} = \dfrac{\lambda}{2\pi\varepsilon_0 s}\hat{s}}$

example: infinite plane with uniform surface charge σ.



Draw Gaussian as a box of height h above & below the surface, and surface area A of upper/lower areas.

By symmetry, $\vec{E}$ must point in $\pm\hat{z}$ direction.



$\vec{E}$ is const & in same direction as $d\vec{A}$ on upper & lower surfaces.

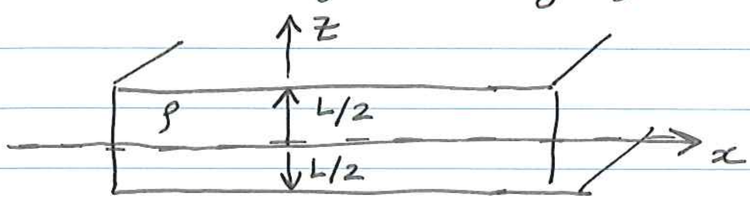
$$\Phi_E = \int d\vec{A} \cdot \vec{E} = 2 A E(z)$$

↑ both upper/lower

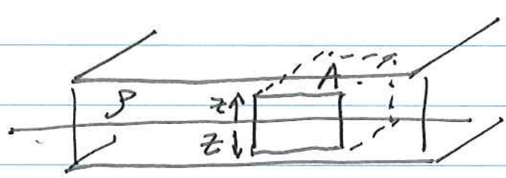
$$\Phi_E = 2AE = \frac{Q_{encl}}{\varepsilon_0} = \frac{1}{\varepsilon_0} \sigma A$$

$$\Rightarrow \quad E = \frac{\sigma}{2\varepsilon_0} \qquad \vec{E} = \begin{cases} \frac{\sigma}{2\varepsilon_0} \hat{z} & z > 0 \\ -\frac{\sigma}{2\varepsilon_0} \hat{z} & z < 0 \end{cases}$$

example: infinite thick slab of thickness L and uniform charge density ρ.

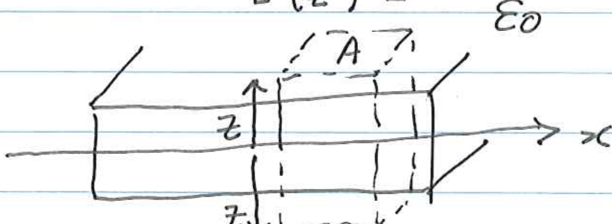


Gaussian surface: box of height $2z$ and upper/lower area A.



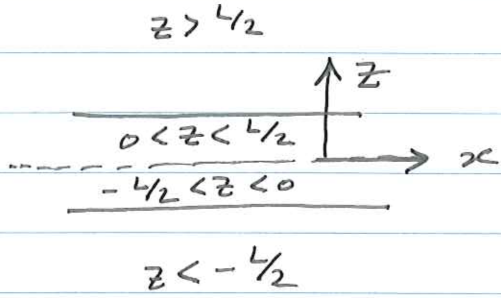
$$\Phi_E = 2AE(z) = \frac{Q_{encl}}{\varepsilon_0} = \frac{1}{\varepsilon_0} \rho \, 2z \, A$$

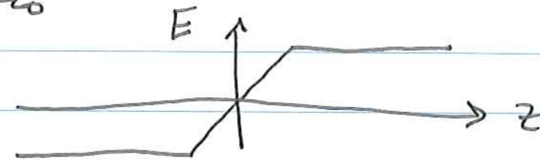
$$E(z) = \frac{\rho z}{\varepsilon_0} \qquad \text{for } 0 < z < L/2$$



$$E(z) = \frac{\rho L}{2\varepsilon_0}$$

Final result for $\vec{E}(z) = \begin{cases} \dfrac{\rho L}{2\varepsilon_0}\hat{z} & z > L/2 \\ \dfrac{\rho z}{2\varepsilon_0}\hat{z} & 0 < z < L/2 \\ -\dfrac{\rho z}{2\varepsilon_0}\hat{z} & -\frac{L}{2} < z < 0 \\ -\dfrac{\rho L}{2\varepsilon_0}\hat{z} & z < -\frac{L}{2} \end{cases}$





Note: outside the slab, looks like an infinite thin plane of surface charge

$$\sigma = \frac{\text{charge}}{\text{area}} = \underbrace{\frac{\text{charge}}{\text{area} \times \text{length}}}_{\rho} \times \underbrace{\text{length}}_{L} = \rho L$$