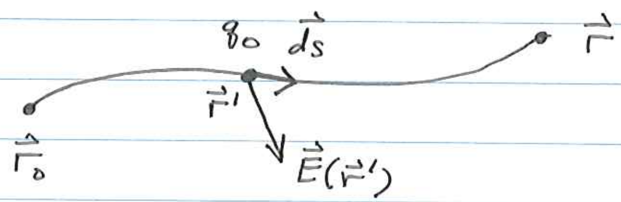
II. Electric potential

Consider a test charge q_0 in an electric field $\vec{E}$.
The force acting on q_0 is $\vec{F} = q_0 \vec{E}$.

What is the work required to move q_0 from an initial position $\vec{r}_0$ to an arbitrary final position $\vec{r}$?

$$W = -\int_{\vec{r}_0}^{\vec{r}} d\vec{s} \cdot \vec{F}(\vec{r}') = -q_0 \int_{\vec{r}_0}^{\vec{r}} d\vec{s} \cdot \vec{E}(\vec{r}')$$



Let $\vec{r}'$ $= (x', y', z')$ be position along path from $\vec{r}_0$ to $\vec{r}$.
$d\vec{s} = (dx', dy', dz')$ points along path.

The electric potential is defined as $\boxed{\phi(\vec{r}) = -\int_{\vec{r}_0}^{\vec{r}} d\vec{s} \cdot \vec{E}(\vec{r}')}$

$$W = q_0 \, \phi(\vec{r})$$

$\phi(\vec{r}) = \dfrac{W}{q_0}$ = Work-per-charge to move a test charge from $\vec{r}_0$ to $\vec{r}$.

Since $\vec{F}$ is conservative for electrostatic forces, $\phi(\vec{r})$ does not depend on path.

Note: $\phi(\vec{r})$ is defined with respect to a <u>reference point</u> $\vec{r}_0$ where $\phi(\vec{r}_0) = 0$.

Defining a "new" potential with respect to a different reference point $\vec{r}_1$ is same as adding a constant.

$$\phi_{new}(\vec{r}) = -\int_{\vec{r}_1}^{\vec{r}} d\vec{s}\cdot\vec{E} = -\int_{\vec{r}_0}^{\vec{r}} d\vec{s}\cdot\vec{E} - \underbrace{\int_{\vec{r}_1}^{\vec{r}_0} d\vec{s}\cdot\vec{E}}_{\text{const. in } \vec{r}.}$$

$$= \phi_{old}(\vec{r}) + \text{constant}$$

Usual convention: $\vec{r}_0 \to \infty$. $\phi = 0$ for $r \to \infty$.

$$\phi(\vec{r}) = -\int_{\infty}^{\vec{r}} d\vec{s}\cdot\vec{E}(\vec{r}\,')$$

$\Delta U(\vec{r}) = q_0\,\phi(\vec{r})$ = change in potential energy by bringing in test charge q_0 from ∞ to position $\vec{r}$.

The <u>potential difference</u> ϕ_{21} between two points $\vec{r}_1$ and $\vec{r}_2$ is:

$$\phi_{21} = \phi(\vec{r}_2) - \phi(\vec{r}_1) = -\int_{\vec{r}_0}^{\vec{r}_2} d\vec{s}\cdot\vec{E} + \int_{\vec{r}_0}^{\vec{r}_1} d\vec{s}\cdot\vec{E}$$

$$= -\int_{\vec{r}_0}^{\vec{r}_2} d\vec{s}\cdot\vec{E} - \int_{\vec{r}_1}^{\vec{r}_0} d\vec{s}\cdot\vec{E} = -\int_{\vec{r}_1}^{\vec{r}_2} d\vec{s}\cdot\vec{E}$$

$W_{21} = q_0\,\phi_{21} = -q_0 \int_{\vec{r}_1}^{\vec{r}_2} d\vec{s}\cdot\vec{E}$ = work to move q_0 from $\vec{r}_1$ to $\vec{r}_2$.

Note: any potential difference is independent of reference point $\vec{r}_0$.

<u>example:</u> point charge q. What is $\phi(\vec{r})$?

Recall potential energy formula for two point charges

$$U_{12} = \frac{1}{4\pi\varepsilon_0} \frac{q_1 q_2}{r_{12}}$$

Here consider q with test charge q_0, separated by distance r.

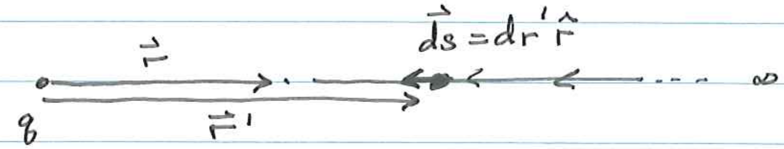
So $q_1 \to q$, $q_2 \to q_0$, $r_{12} \to r$.

$$U_{12} = \frac{1}{4\pi\varepsilon_0} \frac{q q_0}{r} = \text{work to bring in } q_0 \text{ from } \infty$$

$$\text{So} \quad \underline{\phi(r) = \frac{U_{12}}{q_0} = \frac{q}{4\pi\varepsilon_0 r}} \quad \Big\} \text{ doesn't depend on } q_0 \text{ (testcharge)}$$

Or, can compute $\phi(r)$ from $\vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \frac{q}{r^2} \hat{r}$

$$\phi(r) = -\int_{\infty}^{r} \vec{ds} \cdot \vec{E}(\vec{r}\,') \qquad \vec{ds} = dr' \hat{r}$$



$$= -\frac{q}{4\pi\varepsilon_0} \int_{\infty}^{r} dr' \frac{1}{r'^2} = \frac{q}{4\pi\varepsilon_0 r}$$

Potential of a point charge $\boxed{\phi(r) = \frac{q}{4\pi\varepsilon_0 r}}$

Electric potential vs. potential energy

Consider an arbitrary charge configuration, say N point charges $q_1, q_2, \cdots, q_N$, with electric field $\vec{E}$.

The potential energy U_N of the N charges is the total work required to assemble them from ∞.

$$U_N = \sum_{i=1}^{N} \sum_{j=i+1}^{N} U_{ij} = \sum_{i=1}^{N} \sum_{j=i+1}^{N} \frac{q_i q_j}{4\pi\varepsilon_0 r_{ij}}$$

The potential ϕ is related to the extra work it would require to bring in an extra charge q_0.

$$\Delta U = q_0 \phi = \sum_{i=1}^{N} \frac{q_0 q_i}{4\pi\varepsilon_0 r_{0i}}$$

The total potential energy of the $N+1$ charges ($q_1, \cdots, q_N + q_0$) would be $U_{N+1} = U_N + \Delta U$.

example: charged spherical shell with surface charge σ and radius R.

Total charge is $Q = \sigma \, 4\pi R^2$

Inside sphere: $\vec{E} = 0$ by Gauss's law $(r < R)$

Outside sphere: $\vec{E} = \frac{1}{4\pi\varepsilon_0} \frac{Q}{r^2} \hat{r}$ $(r > R)$ same as an equivalent point charge.

For $r > R$: $\phi(r) = \frac{Q}{4\pi\varepsilon_0 r}$ (same as point charge)

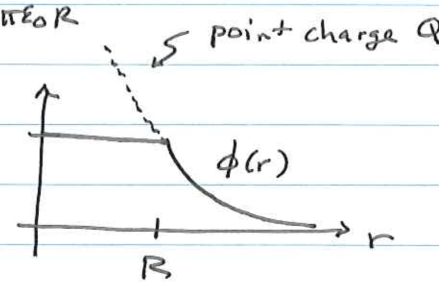
Note: ϕ has SI units $\frac{\text{energy}}{\text{charge}} = \frac{\text{Joule}}{\text{coulomb}} = \text{Volts}$

For $r<R$: $\phi(r) = -\int_{\infty}^{r} d\vec{s}\cdot\vec{E}(\vec{r}') = -\int_{\infty}^{r} dr'\, E(r')$

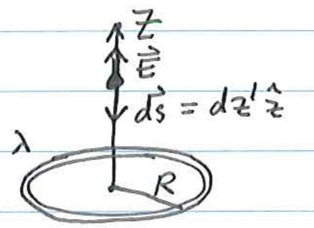
$$= -\int_{\infty}^{R} dr'\, E(r') - \int_{R}^{r} dr'\, E(r')$$

$$= -\int_{\infty}^{R} dr'\, \frac{Q}{4\pi\varepsilon_0 r'^2} = \frac{Q}{4\pi\varepsilon_0 R}$$

So $\phi(r) = \begin{cases} \dfrac{\sigma R^2}{\varepsilon_0 r} & r>R \\ \dfrac{\sigma R}{\varepsilon_0} & r<R \end{cases}$



<u>example:</u> charged ring with linear charge density λ and radius R. What is $\phi(z)$ along z axis?

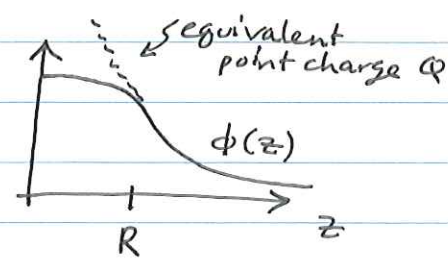


HW #2: $\vec{E}(z) = \dfrac{\lambda R z}{2\varepsilon_0 (R^2+z^2)^{3/2}}\hat{z}$

$$\phi(z) = -\int_{\infty}^{z} d\vec{s}\cdot\vec{E}(z') = -\int_{\infty}^{z} dz'\, \frac{\lambda R z'}{2\varepsilon_0 (R^2+z'^2)^{3/2}} = \frac{\lambda R}{2\varepsilon_0\sqrt{R^2+z^2}}$$

Note the total charge is $Q = \lambda\cdot 2\pi R \Rightarrow \lambda = \dfrac{Q}{2\pi R}$.

$$\phi(z) = \frac{Q}{4\pi\varepsilon_0\sqrt{R^2+z^2}}$$



Everything with nonzero net charge Q looks like a point charge sufficiently far away.

Gradient & partial derivatives

We can compute ϕ from $\vec{E}$ using line integral

$$\phi = -\int d\vec{s} \cdot \vec{E}$$

We can also compute $\vec{E}$ from ϕ using gradient.

Partial derivatives: consider a multivariable function $f(x,y,z)$. The partial derivative $\frac{\partial f}{\partial x}$ is the derivative of f with respect to x, while treating ~~keeping~~ the other variables ~~fixed~~ as constants

e.g. $f(x,y,z) = x^2 y + x$

$$\frac{\partial f}{\partial x} = 2xy + 1$$

$$\frac{\partial f}{\partial y} = x^2$$

$$\frac{\partial f}{\partial z} = 0$$

e.g. $f(x,y,z) = \frac{1}{r} = \frac{1}{\sqrt{x^2+y^2+z^2}}$

$$\frac{\partial f}{\partial x} = -\frac{1}{2}\frac{1}{(x^2+y^2+z^2)^{3/2}}\, 2x = -\frac{x}{r^3}$$

$$\frac{\partial f}{\partial y} = -\frac{y}{r^3}$$

$$\frac{\partial f}{\partial z} = -\frac{z}{r^3}$$

Taylor expansion:

Recall a function $f(x)$ can be Taylor expanded around $x=0$:

$$f(x) = f(0) + f'(0)\,x + \frac{1}{2!} f''(0)\,x^2 + \cdots$$

where $f'(0) = \frac{df}{dx}\Big|_{x=0}$, etc.

or around $x = x_0$ as:

$$f(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{1}{2!} f''(x_0)(x-x_0)^2 + \cdots$$

Using Taylor expansion, $f(x+\Delta x)$ can be related to $f(x)$:

$$f(x+\Delta x) = f(x) + f'(x)\,\Delta x + \frac{1}{2!} f''(x)\,\Delta x^2 + \cdots$$

For Δx small, $f(x+\Delta x) = f(x) + f'(x)\,\Delta x + \cdots$

For multivariable functions, we need to use partial derivatives for Taylor expansion.

$$f(x+\Delta x, y, z) = f(x,y,z) + \frac{\partial f}{\partial x}\,\Delta x + \cdots$$

$$f(x, y+\Delta y, z) = f(x,y,z) + \frac{\partial f}{\partial y}\,\Delta y + \cdots$$

$$f(x, y, z+\Delta z) = f(x,y,z) + \frac{\partial f}{\partial z}\,\Delta z + \cdots$$

More general case:

$$f(x+\Delta x, y+\Delta y, z+\Delta z) = f(x,y,z) + \frac{\partial f}{\partial x}\Delta x + \frac{\partial f}{\partial y}\Delta y + \frac{\partial f}{\partial z}\Delta z + \ldots$$

Let's define $\vec{r} = (x,y,z)$, $\vec{\Delta r} = (\Delta x, \Delta y, \Delta z)$.

Introduce new operator "del" $\vec{\nabla} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$

$$\vec{\nabla} = \frac{\partial}{\partial x}\hat{x} + \frac{\partial}{\partial y}\hat{y} + \frac{\partial}{\partial z}\hat{z}$$

The <u>gradient</u> of f is defined as

$$\vec{\nabla} f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$$

f is a <u>scalar</u> function (a number, no direction)
But $\vec{\nabla} f$ is a <u>vector</u> function.

Note: $\vec{\nabla} f \cdot \vec{\Delta r} = \frac{\partial f}{\partial x}\Delta x + \frac{\partial f}{\partial y}\Delta y + \frac{\partial f}{\partial z}\Delta z$

So we can write:

$$\underbrace{f(\vec{r} + \vec{\Delta r})}_{f(x+\Delta x,\, y+\Delta y,\, z+\Delta z)} = \underbrace{f(\vec{r})}_{f(x,y,z)} + \vec{\nabla} f \cdot \vec{\Delta r} + \ldots$$

We can use gradient to compute $\vec{E}$ from ϕ:

Consider the potential difference $\Delta\phi$ between ϕ at $\vec{r}$ and $\vec{r}+\vec{\Delta r}$.

$$\Delta\phi = \phi(\vec{r}+\vec{\Delta r}) - \phi(\vec{r})$$

$$= -\int_{\vec{r}_0}^{\vec{r}+\vec{\Delta r}} \vec{ds}\cdot\vec{E}(\vec{r}\,') + \int_{\vec{r}_0}^{\vec{r}} \vec{ds}\cdot\vec{E}(\vec{r}\,') = -\int_{\vec{r}}^{\vec{r}+\vec{\Delta r}} \vec{ds}\cdot\vec{E}(\vec{r}\,')$$

Assume $\vec{\Delta r}$ is small.

Then $\Delta\phi = \phi(\vec{r}+\vec{\Delta r}) - \phi(\vec{r}) = \phi(\vec{r}) + \vec{\nabla}\phi\cdot\vec{\Delta r} - \phi(\vec{r})$

$$= \vec{\nabla}\phi\cdot\vec{\Delta r}$$

Also, can assume $\vec{E}$ is approx. const along path from $\vec{r}$ to $\vec{r}+\vec{\Delta r}$.

$$\int_{\vec{r}}^{\vec{r}+\vec{\Delta r}} \vec{ds}\cdot\vec{E}(\vec{r}\,') = \vec{E}(\vec{r})\cdot\left(\int_{\vec{r}}^{\vec{r}+\vec{\Delta r}} \vec{ds}\right) = \vec{E}(\vec{r})\cdot\vec{\Delta r}$$

Note: $\displaystyle\int_{\vec{r}}^{\vec{r}+\vec{\Delta r}} \vec{ds} = \int_{(x,y,z)}^{(x+\Delta x,\, y+\Delta y,\, z+\Delta z)} (dx', dy', dz') = (\Delta x, \Delta y, \Delta z)$

For small $\vec{\Delta r}$ we have:

$$\vec{\nabla}\phi \cdot \vec{\Delta r} = -\vec{E}(\vec{r}) \cdot \vec{\Delta r}$$

This relation must hold for <u>any</u> (small) $\vec{\Delta r}$.

Let $\vec{\Delta r} = (\Delta x, 0, 0)$, displacement in x only.

$$\vec{\nabla}\phi \cdot \vec{\Delta r} = \left(\frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, \frac{\partial \phi}{\partial z}\right) \cdot (\Delta x, 0, 0) = \frac{\partial \phi}{\partial x} \Delta x$$

$$= -\vec{E} \cdot \vec{\Delta r} = -E_x \Delta x$$

$$\Rightarrow \quad E_x = -\frac{\partial \phi}{\partial x}$$

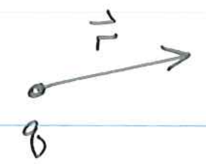
Similarly for other displacements:

$$\vec{\Delta r} = (0, \Delta y, 0) \Rightarrow \quad E_y = -\frac{\partial \phi}{\partial y}$$

$$\vec{\Delta r} = (0, 0, \Delta z) \Rightarrow \quad E_z = -\frac{\partial \phi}{\partial z}$$

So we have: $(E_x, E_y, E_z) = -\left(\frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, \frac{\partial \phi}{\partial z}\right)$

Or in vector form: $\boxed{\vec{E}(\vec{r}) = -\vec{\nabla}\phi(\vec{r})}$



example: point charge q.

Potential is $\phi(\vec{r}) = \dfrac{q}{4\pi\varepsilon_0 r}$. Compute $\vec{E}$.

$$\vec{E} = -\vec{\nabla}\phi = -\frac{q}{4\pi\varepsilon_0}\vec{\nabla}\left(\frac{1}{r}\right)$$

Note: we already computed

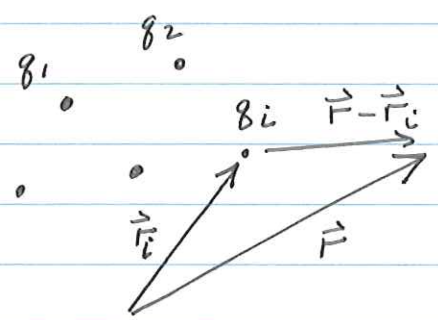
$$\frac{\partial}{\partial x}\left(\frac{1}{r}\right) = -\frac{x}{r^3},\quad \frac{\partial}{\partial y}\left(\frac{1}{r}\right) = -\frac{y}{r^3},\quad \frac{\partial}{\partial z}\left(\frac{1}{r}\right) = -\frac{z}{r^3}$$

Then $\vec{\nabla}\left(\frac{1}{r}\right) = -\left(\frac{x}{r^3}, \frac{y}{r^3}, \frac{z}{r^3}\right) = -\frac{1}{r^3}(x,y,z)$

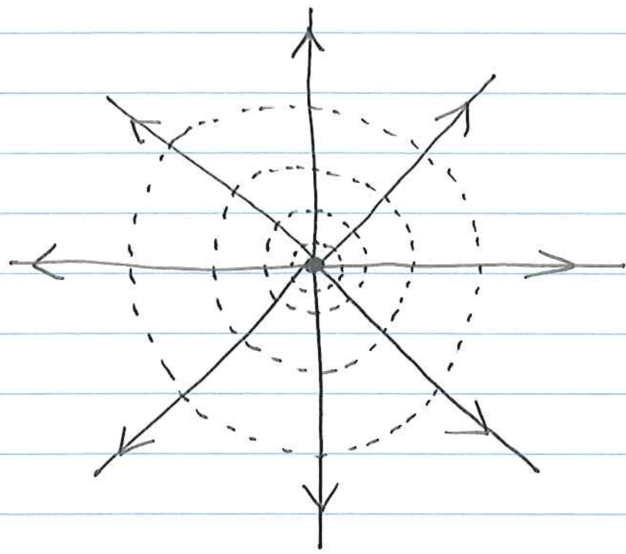
$$= -\frac{1}{r^3}\vec{r} = -\frac{\hat{r}}{r^2}$$

$$\underline{\vec{E}(\vec{r}) = \frac{q}{4\pi\varepsilon_0}\frac{\hat{r}}{r^2}}$$

example: multiple point charges $q_1, q_2, \cdots q_N$ at positions $\vec{r}_1, \vec{r}_2, \ldots$ What is ϕ at $\vec{r}$?



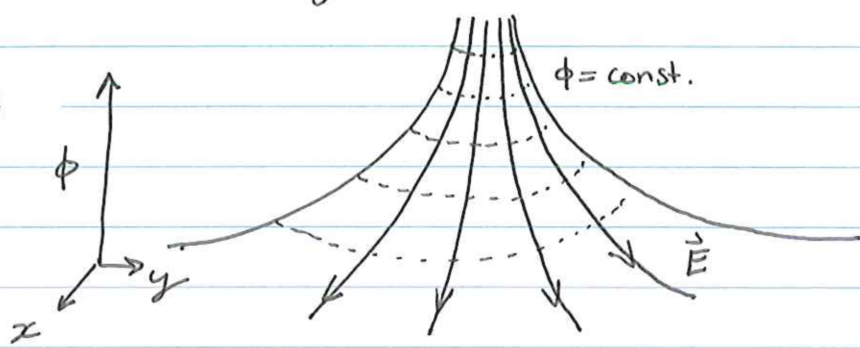
$$\phi(\vec{r}) = \sum_{i=1}^{N}\frac{q_i}{4\pi\varepsilon_0|\vec{r}-\vec{r}_i|}$$



Electric field lines from point charge

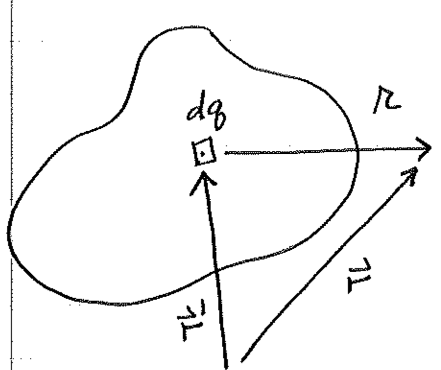
Lines of constant $\phi(\vec{r})$
= <u>equipotentials</u>

ϕ is like the height of ~~the~~ a hillside.
Equipotentials are contours of constant height along the hill.



$\vec{E} = -\vec{\nabla}\phi$ points down hill along steepest path, perpendicular to equipotentials.

Potential for continuous charge distributions



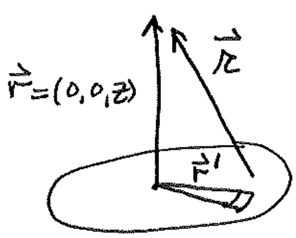
$\vec{r}'$ = position of dq
$\vec{r}$ = where ϕ is measured
$\vec{\mathscr{r}} = \vec{r} - \vec{r}'$

$$\phi(\vec{r}) = \sum_{dq} \frac{dq}{4\pi\varepsilon_0 \mathscr{r}} \rightarrow \int \frac{dq}{4\pi\varepsilon_0 \mathscr{r}}$$

e.g. charge density $\rho \Rightarrow dq = \rho\, dV$

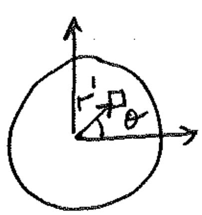
$$\phi(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \int dV \frac{\rho(\vec{r}')}{\mathscr{r}} = \frac{1}{4\pi\varepsilon_0} \int dV \frac{\rho(\vec{r}')}{|\vec{r} - \vec{r}'|}$$

<u>example</u>: charged disk of radius R and surface charge σ.
what is potential ϕ at $\vec{r} = (0,0,z)$



$\vec{r}' = (x', y', 0)$ ← write in spherical coords

$= (s\cos\theta, s\sin\theta, 0)$



$\vec{r} = (0,0,z)$

$\vec{\mathscr{r}} = (-s\cos\theta, -s\sin\theta, z)$

$\mathscr{r} = (s^2 + z^2)^{1/2}$

$$\phi(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \int \frac{dA}{r} = \frac{1}{4\pi\varepsilon_0} \int \frac{\sigma \, s \, ds \, d\theta}{\sqrt{s^2+z^2}}$$

$$= \frac{\sigma}{4\pi\varepsilon_0} \int_0^R \frac{s\,ds}{\sqrt{s^2+z^2}} \int_0^{2\pi} d\theta$$

$$= \frac{\sigma}{4\pi\varepsilon_0} 2\pi \cdot \left(\sqrt{R^2+z^2} - z \right)$$

$$\phi(z) = \frac{\sigma}{2\varepsilon_0} \left(\sqrt{R^2+z^2} - z \right)$$

Check: $\vec{E} = -\vec{\nabla}\phi$

$$E_z = -\frac{\partial \phi}{\partial z} = \frac{\sigma}{2\varepsilon_0} \left(1 - \frac{z}{\sqrt{z^2+R^2}} \right)$$

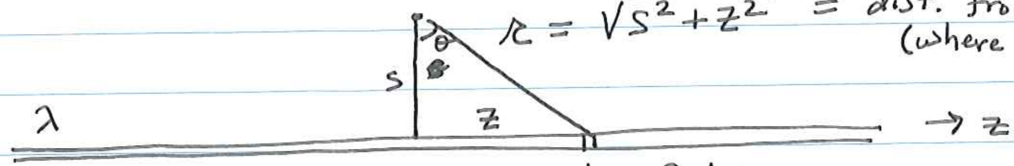
$E_x = E_y = 0$ by symmetry.

$$\Rightarrow \quad \vec{E}(0,0,z) = \frac{\sigma}{2\varepsilon_0} \left(1 - \frac{z}{\sqrt{z^2+R^2}} \right) \hat{z}$$

↑ Tutorial #2

example: infinite wire. Find ϕ a distance s from wire.

$\mathscr{r} = \sqrt{s^2+z^2}$ = dist. from dq to $\vec{r}$ (where ϕ is measured)



$dq = \lambda\, dz$

$$\phi = \int \frac{dq}{4\pi\varepsilon_0 \mathscr{r}} = \int_{-\infty}^{\infty} \frac{\lambda\, dz}{4\pi\varepsilon_0 (s^2+z^2)^{1/2}}$$

Trig. substitution: $z = s\tan\theta$

$dz = s \sec^2\theta\, d\theta$

$z = \pm\infty \longrightarrow \theta = \pm\frac{\pi}{2}$

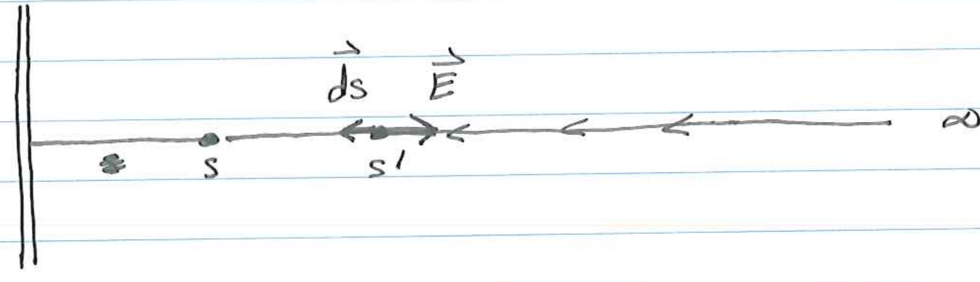
$$\phi = \frac{\lambda}{4\pi\varepsilon_0} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta \, \frac{\sec^2\theta}{\sec\theta} = \frac{\lambda}{4\pi\varepsilon_0} \underbrace{\int_{-\frac{\pi}{2}}^{\pi/2} d\theta \, \sec\theta}_{=\infty}$$

We get $\phi(s) = \infty$. What went wrong here?

Compute ϕ from line integral of $\vec{E}$:

$$\vec{E} = \frac{\lambda}{2\pi\varepsilon_0 s}\hat{s}$$

Compute ϕ by integrating along $\hat{s}$ direction



s' is position along path from ∞ to final position s. $\vec{ds} = ds'\,\hat{s}$

$$\phi(s) = -\int_{\infty}^{s} ds'\,\hat{s}\cdot\vec{E}(s') = -\int_{\infty}^{s} ds'\,\frac{\lambda}{2\pi\varepsilon_0 s'}$$

$$= \frac{\lambda}{2\pi\varepsilon_0}\int_{s}^{\infty} ds'\,\frac{1}{s'} = \frac{\lambda}{2\pi\varepsilon_0}\ln s'\Big|_{s'=s}^{s'=\infty} = \infty$$

Because we defined the potential with respect to position $\vec{r}_0 \to \infty$, we got $\phi = \infty$.

For infinite charge distributions, it takes an infinite amount of work to bring in a test charge q_0 from ∞.

Still can define potential, but with respect to a finite $\vec{r}_0$.

e.g. Suppose $\vec{r}_0$ is a distance s_0 from the wire, then

$$\phi(s) = \frac{\lambda}{2\pi\varepsilon_0}\ln s'\Big|_{s}^{s_0} = \frac{\lambda}{2\pi\varepsilon_0}\ln(s_0/s)$$

For a finite charge distribution (i.e. localized within some finite volume), can define ϕ with respect to $\vec{r}_0 \to \infty$.

Note: $\vec{E}$ doesn't depend on $\vec{r}_0$.

In Cartesian coords: $s = \sqrt{x^2+y^2}$

So $$\phi(x,y,z) = \frac{\lambda}{2\pi\varepsilon_0} \ln\left(\frac{s_0}{\sqrt{x^2+y^2}}\right)$$

$$= \frac{\lambda}{2\pi\varepsilon_0}\left(\ln s_0 - \frac{1}{2}\ln(x^2+y^2)\right)$$

Then

$$\vec{E} = -\vec{\nabla}\phi = -\left(\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, 0\right)$$

$$= \frac{\lambda}{4\pi\varepsilon_0}\frac{1}{x^2+y^2}(2x, 2y, 0)$$

$$= \frac{\lambda}{2\pi\varepsilon_0 s}\underbrace{\left(\frac{x}{s}, \frac{y}{s}, 0\right)}_{\hat{s} = \frac{x}{s}\hat{x} + \frac{y}{s}\hat{y}} = \frac{\lambda}{2\pi\varepsilon_0 s}\hat{s}$$

Here we computed gradient $\vec{\nabla}\phi$ by converting to Cartesian coordinates (x,y,z) since $\vec{\nabla} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) = \frac{\partial}{\partial x}\hat{x} + \frac{\partial}{\partial y}\hat{y} + \frac{\partial}{\partial z}\hat{z}$

But you can also express $\vec{\nabla}$ in cylindrical or spherical coords as well.

First, recall the chain rule:

Suppose you have a function $f(y)$, but y depends on x according to $y(x)$.

You can express f as a function of x as:

$$f(y(x))$$

What is the derivative with respect to x?

$$\frac{df}{dx} = \frac{d}{dx} f(y(x)) = f'(y(x)) \cdot y'(x)$$

$$\frac{df}{dx} = \frac{df}{dy} \cdot \frac{dy}{dx}$$

Now, suppose $\phi = \phi(s, z)$ only depends on s, z in cylindrical coords. Treat $s = s(x,y) = \sqrt{x^2+y^2}$ as a function of x, y.

$$\vec{\nabla} \phi(s,z) = \vec{\nabla} \phi(s(x,y), z)$$

$$= \left(\frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, \frac{\partial \phi}{\partial z} \right)$$

$$= \left(\frac{\partial \phi}{\partial s} \frac{\partial s}{\partial x}, \frac{\partial \phi}{\partial s} \frac{\partial s}{\partial y}, \frac{\partial \phi}{\partial z} \right)$$ chain rule

$$= \left(\frac{x}{s} \frac{\partial \phi}{\partial s}, \frac{y}{s} \frac{\partial \phi}{\partial s}, \frac{\partial \phi}{\partial z} \right)$$

$$= \frac{\partial \phi}{\partial s} \underbrace{\left(\frac{x}{s} \hat{x} + \frac{y}{s} \hat{y} \right)}_{\hat{s}} + \frac{\partial \phi}{\partial z} \hat{z}$$

So $\vec{\nabla}\phi(s,z) = \frac{\partial\phi}{\partial s}\hat{s} + \frac{\partial\phi}{\partial z}\hat{z}$

$$\boxed{\vec{\nabla} = \frac{\partial}{\partial s}\hat{s} + \frac{\partial}{\partial z}\hat{z}}$$

Infinite wire example:

$$\vec{E} = -\vec{\nabla}\left(\frac{\lambda}{2\pi\varepsilon_0}\ln\left(s_0/s\right)\right) = -\frac{\partial}{\partial s}\left(\frac{\lambda}{2\pi\varepsilon_0}\ln\left(s_0/s\right)\right)\hat{s}$$

$$= \frac{\lambda}{2\pi\varepsilon_0 s}\hat{s}$$

Spherical coordinates: assume ϕ only depends on $r = r(x,y,z) = \sqrt{x^2+y^2+z^2}$.

Also recall $\hat{r} = \left(\frac{x}{r}, \frac{y}{r}, \frac{z}{r}\right)$.

$$\vec{\nabla}\phi(r) = \left(\frac{\partial\phi}{\partial r}\frac{\partial r}{\partial x}, \frac{\partial\phi}{\partial r}\frac{\partial r}{\partial y}, \frac{\partial\phi}{\partial r}\frac{\partial r}{\partial z}\right)$$

$$= \frac{\partial\phi}{\partial r}\left(\frac{x}{r}, \frac{y}{r}, \frac{z}{r}\right) = \frac{\partial\phi}{\partial r}\hat{r}$$

So $\boxed{\vec{\nabla} = \frac{\partial}{\partial r}\hat{r}}$

Gradient:

Cartesian	$\vec{\nabla} = \frac{\partial}{\partial x}\hat{x} + \frac{\partial}{\partial y}\hat{y} + \frac{\partial}{\partial z}\hat{z}$
Cylindrical	$\vec{\nabla} = \frac{\partial}{\partial s}\hat{s} + \frac{\partial}{\partial z}\hat{z}$
Spherical	$\vec{\nabla} = \frac{\partial}{\partial r}\hat{r}$

only valid when ϕ doesn't depend on angles θ or (θ,ϕ) otherwise more complicated.

example: point charge $\phi(r) = \dfrac{q}{4\pi\varepsilon_0 r}$

$$\vec{E} = -\vec{\nabla}\phi = -\hat{r}\,\frac{\partial\phi}{\partial r} = \frac{q}{4\pi\varepsilon_0 r^2}\,\hat{r}$$

Electric dipole moments & multipole expansion

Any finite charge distribution with total charge Q and typical size R (e.g. ring of radius R) behaves likes a point charge Q sufficiently far away ($r \gg R$): $\phi(r) = \dfrac{Q}{4\pi\varepsilon_0 r}$ for $r \gg R$.

This idea can be generalized using multipole expansion

$\phi(r) = \dfrac{Q}{4\pi\varepsilon_0 r}$ is called the monopole term. Depends on total charge Q and falls as $1/r$.

Consider a configuration where $Q = 0$.
Two charges $+q$ and $-q$ separated by distance d.
Find $\phi(\vec{r})$ for $r \gg d$.