example: point charge $\phi(r) = \frac{q}{4\pi\varepsilon_0 r}$

$$\vec{E} = -\vec{\nabla}\phi = -\hat{r}\frac{\partial\phi}{\partial r} = \frac{q}{4\pi\varepsilon_0 r^2}\hat{r}$$

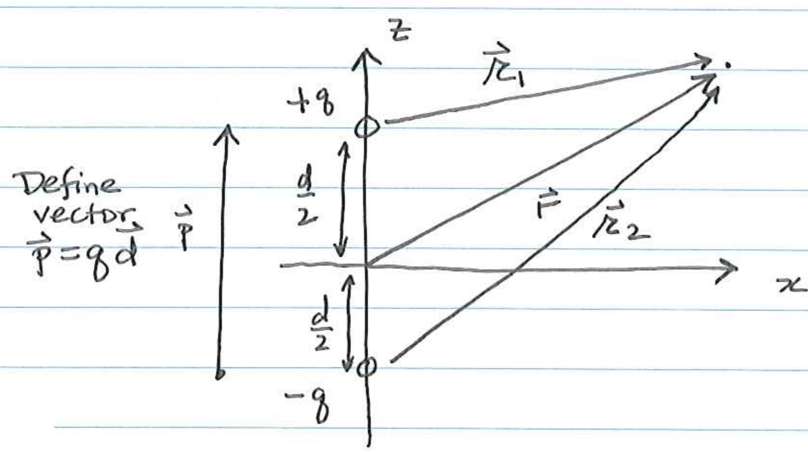
Electric dipole moments & multipole expansion

Any finite charge distribution with total charge Q and typical size R (e.g. ring of radius R) behaves likes a point charge Q sufficiently far away ($r \gg R$): $\phi(r) = \frac{Q}{4\pi\varepsilon_0 r}$ for $r \gg R$.

This idea can be generalized using multipole expansion

$\phi(r) = \frac{Q}{4\pi\varepsilon_0 r}$ is called the monopole term. Depends on total charge Q and falls as $1/r$.

Consider a configuration where $Q = 0$.
Two charges $+q$ and $-q$ separated by distance d.
Find $\phi(\vec{r})$ for $r \gg d$.



$$\phi(\vec{r}) = \frac{q}{4\pi\varepsilon_0 \mathscr{r}_1} + \frac{-q}{4\pi\varepsilon_0 \mathscr{r}_2}$$

$$\vec{\mathscr{r}}_1 = (x, y, z - \tfrac{d}{2}) \rightarrow \mathscr{r}_1 = \sqrt{x^2 + y^2 + (z - \tfrac{d}{2})^2}$$

$$\vec{\mathscr{r}}_2 = (x, y, z + \tfrac{d}{2}) \rightarrow \mathscr{r}_2 = \sqrt{x^2 + y^2 + (z + \tfrac{d}{2})^2}$$

$$\frac{1}{\mathscr{r}_{1,2}} = \frac{1}{\sqrt{x^2 + y^2 + z^2 \mp z d + (\frac{d^2}{4})}} = \frac{1}{\sqrt{r^2 \mp zd + d^2/4}}$$

$$= \frac{1}{r} \frac{1}{\sqrt{1 \mp \frac{zd}{r^2} + \frac{d^2}{4r^2}}}$$, note also $z = r\cos\theta$

We want to consider $d \ll r$. Taylor expand in d:

$$f(d) = \frac{1}{\sqrt{1 \mp \frac{d}{r}\cos\theta + \frac{d^2}{4r^2}}} = f(0) + f'(0)\, d + \ldots$$

$$= 1 + (-\tfrac{1}{2})(\mp \tfrac{\cos\theta}{r})\, d + \ldots$$

$$= 1 \pm \frac{dz}{2r^2} + \ldots$$

So we have:

$$\phi(\vec{r}) = \frac{q}{4\pi\varepsilon_0 r}\left(1 + \frac{dz}{2r^2}\right) - \frac{q}{4\pi\varepsilon_0 r}\left(1 - \frac{dz}{2r^2}\right) + \ldots$$ drop smaller terms

$$= \frac{q d z}{4\pi\varepsilon_0 r^3} = \frac{q d \cos\theta}{4\pi\varepsilon_0 r^2} = \frac{p\cos\theta}{4\pi\varepsilon_0 r^2}$$

Define <u>electric dipole moment</u> $p = qd$

Multipole expansion:

Monopole term: $\phi(\vec{r}) \sim \frac{Q_{tot}}{r}$, usually dominant for large r.

Here $Q_{tot} = q - q = 0$.

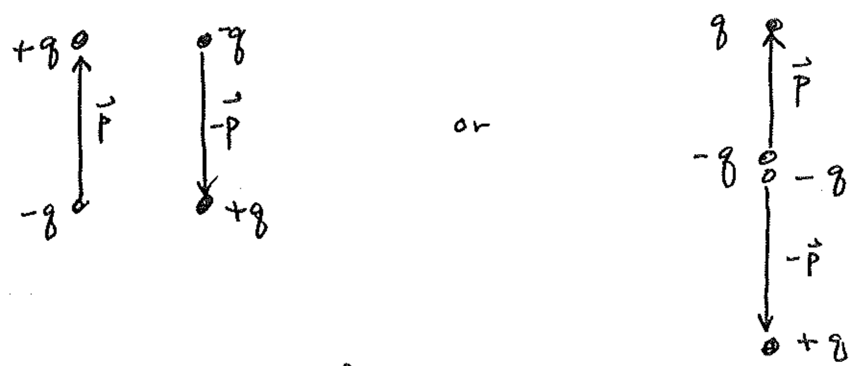
Dipole term: $\phi(\vec{r}) \sim \frac{qd}{r^2} = \frac{p}{r^2}$

usually dominant for large r when $Q_{tot} = 0$.

Higher order terms: any potential $\phi(\vec{r})$ can be expanded in powers of $\left(\frac{d}{r}\right)^n$, where

d is a distance scale ~ size of charge distribution $\ll r$

Quadrupole term: two dipoles that cancel



$$\phi(\vec{r}) \sim \frac{qd^2}{4\pi\varepsilon_0 r^3}$$

Octopole term: $\phi \sim \frac{qd^3}{4\pi\varepsilon_0 r^4}$

General multipole moment $\phi \sim \frac{qd^n}{4\pi\varepsilon_0 r^{n+1}}$

Since $d \ll r$, term with fewest powers of d/r is dominant, unless it vanishes.

Divergence & curl

We showed that $\vec{\nabla} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$ can operate on a scalar function (e.g. $\phi(\vec{r})$) to give a vector function (e.g. $\vec{E}(\vec{r})$). This is the gradient.

$$\text{"grad } \phi\text{"} = \vec{\nabla}\phi = \left(\frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, \frac{\partial \phi}{\partial z}\right) = -\vec{E}.$$

What happens when $\vec{\nabla}$ operates on a vector function? There are two ways to multiply vectors: dot product & cross product. Two ways for $\vec{\nabla}$ to act on a vector function ($\vec{E}$).

<u>divergence</u>: $$\vec{\nabla}\cdot\vec{E} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)\cdot(E_x, E_y, E_z)$$

$$= \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}$$

<u>curl</u>: $$\vec{\nabla}\times\vec{E} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ E_x & E_y & E_z \end{vmatrix}$$

$$= \left(\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z}\right)\hat{x} + \left(\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x}\right)\hat{y}$$

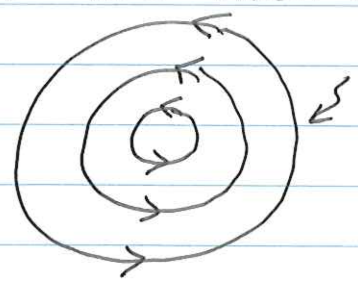
$$+ \left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}\right)\hat{z}$$

$$= \left(\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z}, \frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x}, \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}\right)$$

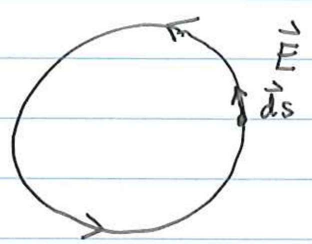
This is in Cartesian coordinates. Can also evaluate divergence & curl in curvilinear coordinates, but it is more complicated. (3020)

Physical interpretation for electrostatics

Curl : $\vec{\nabla} \times \vec{E}(\vec{r})$ how much $\vec{E}$ rotates around in a loop



If $\vec{E}$ forms a loop, it takes nonzero work to bring a test charge go around the loop.



Let C be the closed loop.
$\vec{ds}$ points in same direction as $\vec{E}$

$$W = -q_0 \int_C \vec{ds} \cdot \vec{E} < 0.$$

Violates rule that electrostatic forces are conservative.
So we must have $\boxed{\vec{\nabla} \times \vec{E} = 0}$

Check: $$\vec{\nabla} \times \vec{E} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ -\partial\phi/\partial x & -\partial\phi/\partial y & \partial\phi/\partial z \end{vmatrix} = -\vec{\nabla} \times (\vec{\nabla}\phi)$$

$$= \left(\frac{\partial^2 \phi}{\partial y \partial z} - \frac{\partial^2 \phi}{\partial z \partial y} \right) \hat{x} + \ldots = 0$$

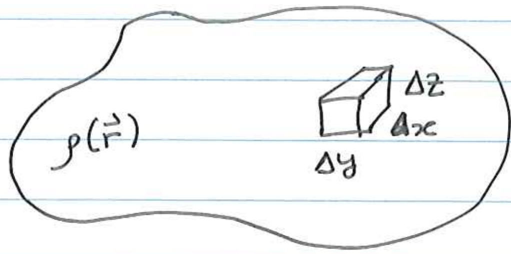
True for any function $\phi(\vec{r})$ since derivatives commute.

$$\frac{\partial}{\partial x}\frac{\partial}{\partial y}\phi = \frac{\partial}{\partial y}\frac{\partial}{\partial x}\phi$$

Divergence: what is $\vec{\nabla} \cdot \vec{E}$?

Consider Gauss's Law $\quad \Phi_E = \int d\vec{A} \cdot \vec{E} = \dfrac{Q_{encl}}{\varepsilon_0}$

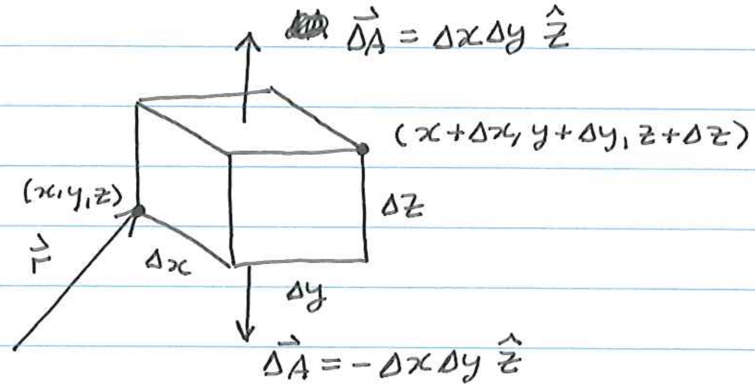
Arbitrary charge density $\rho(\vec{r})$. Let's use Gauss's law for a tiny box of volume $\Delta V = \Delta x \Delta y \Delta z$ located at point $\vec{r} = (x, y, z)$.



$$Q_{encl} = \int_{\Delta V} dV \, \rho(\vec{r}') = \rho(\vec{r}') \int_{\Delta V} dV = \rho(\vec{r}) \, \Delta V$$

if ΔV is small enough, ρ is approx const over ΔV.

Next, compute the flux Φ_E: consider top/bottom surfaces only.



$$\Phi_E^{top} = \vec{E}(x, y, z+\Delta z) \cdot \vec{\Delta A}$$
$$= E_z(x, y, z+\Delta z) \, \Delta x \, \Delta y$$

$$\Phi_E^{bottom} = \vec{E}(x, y, z) \cdot \vec{\Delta A} = -E_z(x, y, z) \, \Delta x \, \Delta y$$

Sum : $\Phi_E^{top} + \Phi_E^{bottom} = \left[E_z(x,y,z+\Delta z) - E_z(x,y,z) \right] \Delta x \Delta y$

$$= \frac{\partial E_z}{\partial z} \Delta z \Delta x \Delta y = \frac{\partial E_z}{\partial z} \Delta V$$

Same for other faces but $z \to x, y$.

$$\Phi_E^{total} = \left(\frac{\partial E_z}{\partial z} + \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} \right) \Delta V = (\vec{\nabla} \cdot \vec{E}) \, \Delta V$$

So we have: $\boxed{\vec{\nabla} \cdot \vec{E}(\vec{r}) = \frac{\rho(\vec{r})}{\varepsilon_0}}$

Maxwell's equations for electrostatics:

Integral form:

$\int d\vec{A} \cdot \vec{E} = \frac{Q_{encl}}{\varepsilon_0}$ (Gauss's law)

$\int_C d\vec{s} \cdot \vec{E} = 0$ for closed loop C

(conservative force)

Differential form:

$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\varepsilon_0}$

$\vec{\nabla} \times \vec{E} = 0$