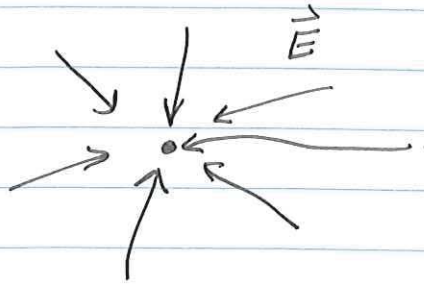
Earnshaw's theorem: It is not possible to trap a point charge q in a region of empty space by electrostatic fields (static $\vec{E}$). i.e. Impossible to have an equilibrium point for q that is stable in every direction.

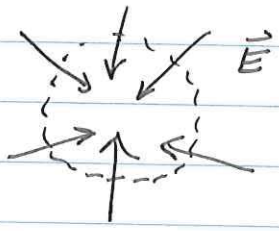
proof: using Gauss's law

If we had any empty region where $\vec{E}$ pointed inward along every direction, then q would be trapped.



this would be a stable equilibrium point since $\vec{F} = q\vec{E}$ points back to original position for any small displacement.

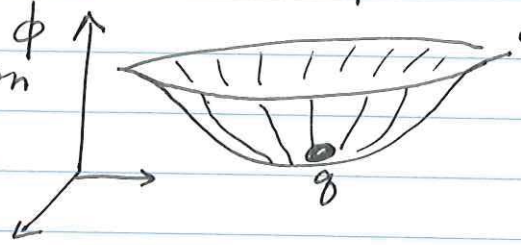
Draw Gaussian surface:



It is clear that $\Phi_E \neq 0$ so we must have $Q_{encl} \neq 0$. This violates assumption that our trap was in empty space with $Q_{encl} = 0$.

proof: using Laplace's equation: $\nabla^2 \phi = 0$

To trap, q we want a potential that has a minimum at some position inside some region.



Not possible.