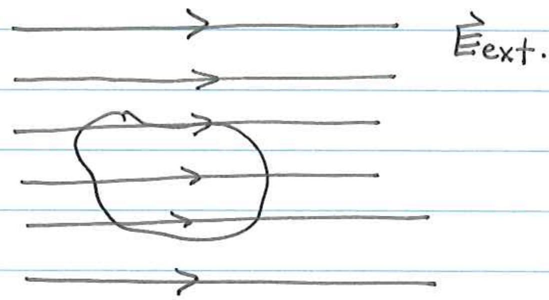
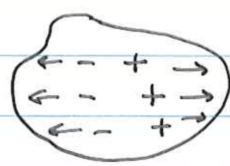
III. Conductors (in electrostatics)

So far we have just considered insulators: charge distribution of an object is fixed

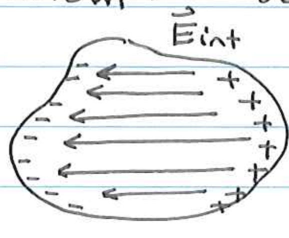
In conductors, charge carriers (electrons) are free to move around. Consider a conducting ~~material~~ body with total charge $Q=0$ inside an external electric field.



Positive & negative charge moves to the right & left, resp., due to $\vec{E}$.



Reaches equilibrium: separation of charges sets up an internal $\vec{E}$ field $\vec{E}_{int}$.

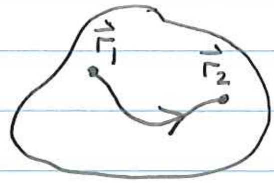


equilibrium reached when total $\vec{E}$ inside conductor is $\vec{E} = \vec{E}_{int} + \vec{E}_{ext} = 0$

So $\vec{E} = 0$ inside a conductor. If $\vec{E} \neq 0$, then charges would continue to move $\rightarrow$ eventually reach equilibrium where $\vec{E} = 0$ and charges are stationary.

Since $\vec{E}=0$, $\rho = \varepsilon_0 \vec{\nabla}\cdot\vec{E} = 0$ inside a conductor.

Zero charge density inside a conductor.

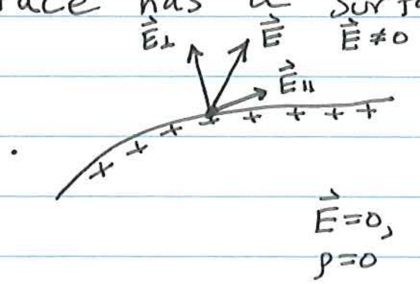


Consider the potential difference between any two points $\vec{r}_1$ and $\vec{r}_2$ in the conductor:

$$\phi_{21} = \phi(\vec{r}_2) - \phi(\vec{r}_1) = -\int_{\vec{r}_1}^{\vec{r}_2} d\vec{s}\cdot\vec{E} = 0 \quad \text{since } \vec{E}=0$$

So all points inside the conductor have the <u>same</u> potential. $\phi =$ constant inside conductor.

Next, consider the surface of the conductor. All charge located on the surface, since $\rho=0$ inside. Surface has a surface charge density.



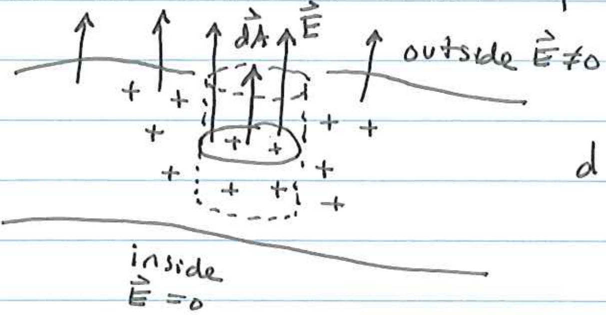
What is $\vec{E}$ on the surface? (just outside the surface)

Write $\vec{E} = \underbrace{\vec{E}_\perp}_{\text{perp. to surface}} + \underbrace{\vec{E}_\parallel}_{\text{parallel to surface}}$. If $\vec{E}_\parallel \neq 0$, then surface will move along the surface.

Eventually charges will reach equilibrium and $\vec{E}_\parallel$ must be zero.

So $\vec{E}$ at conductor surface must be <u>normal</u> to conductor.

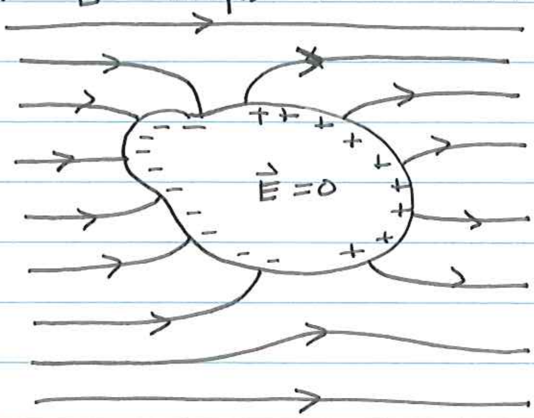
Surface charge σ related to $\vec{E}$ by Gauss's Law.
Zoom in on a small patch $d\vec{A}$:



$$d\Phi_E = \vec{E} \cdot d\vec{A} = E\,dA = \frac{Q_{encl}}{\varepsilon_0}$$
$$= \frac{\sigma\,dA}{\varepsilon_0}$$

So $E = \frac{\sigma}{\varepsilon_0}$ on the conductor (just outside). $\vec{E}$ points normal to the surface.

Total $\vec{E}$ looks like this:



$\vec{E}$ lines.

Rules for conductors:

(1) $\vec{E} = 0$ inside

(2) $\rho = 0$ inside

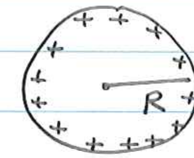
(3) $\phi = \text{const}$ inside

(4) $\vec{E}$ just outside is normal to surface.

(5) $E = \sigma/\varepsilon_0$, where σ = surface density

(6) Total charge on conductor is $Q = \int \sigma\,dA$ over all surfaces.

example: Solid conducting sphere with radius R and total charge Q. What is $\vec{E}$?



Since $\rho = 0$ inside, all charge must reside at the surface. Also must have $\vec{E} = 0$ inside.

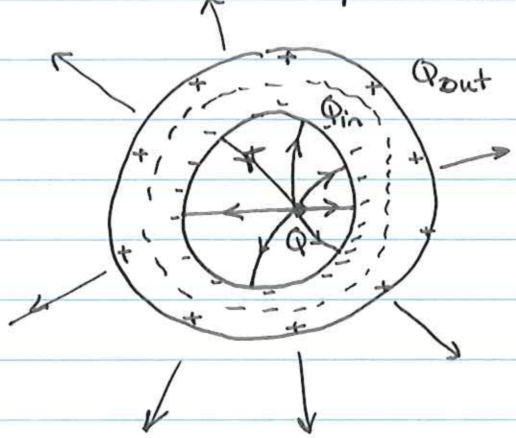
Recall: $\vec{E} = 0$ inside spherical shell of uniform surface charge σ.

$\Rightarrow$ Q is uniformly distributed on surface of conducting sphere with $\sigma = Q/4\pi R^2$

$$\vec{E} = \begin{cases} \dfrac{Q}{4\pi\varepsilon_0 r^2}\hat{r} & r > R \\ 0 & r < R \end{cases}$$ same as shell.

At surface: $E = \dfrac{\sigma}{\varepsilon_0} = \dfrac{Q}{4\pi\varepsilon_0 R^2}$

example: thick, conducting spherical shell, with total charge 0 and a point charge Q in the inner cavity.



Charge Q_{in} on inner surface & Q_{out} on outer surface.

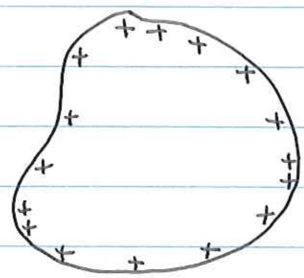
Use Gauss's law on surface through the conductor.

$\vec{E} = 0 \Rightarrow Q_{encl} = 0 \Rightarrow Q_{in} + Q = 0.$

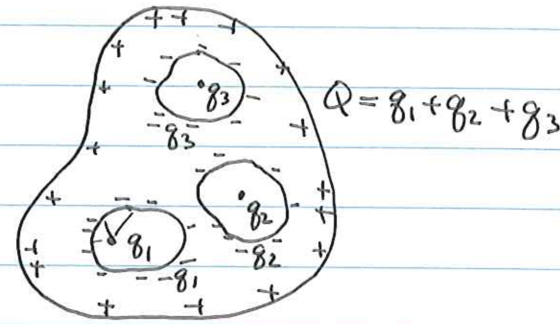
$\Rightarrow Q_{in} = -Q$

Since $Q_{out} + Q_{in} = 0 \Rightarrow Q_{out} = +Q$.
Must be uniformly distributed on outer surface as previous example.

Conductors "hide" any internal distribution of charge.
Same charge distribution on outside surface of conductor:



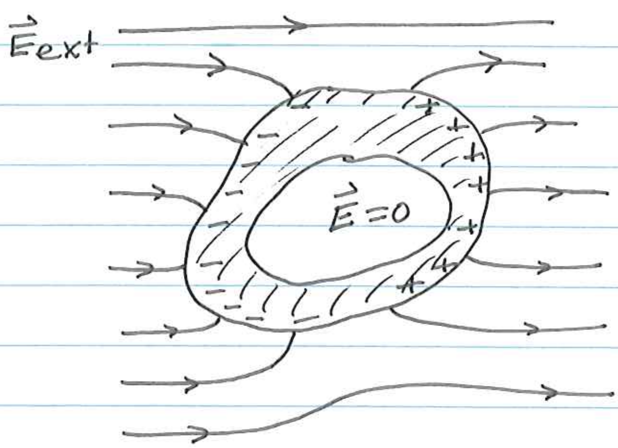
solid conductor with charge Q



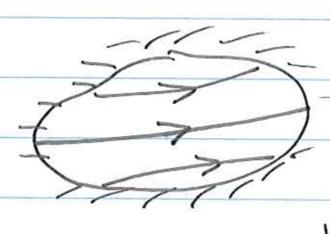
solid conductor with charge 0 hiding point charges q_i, such that $q_1 + q_2 + .. = Q$.

Same external electric field $\vec{E}$ outside conductor.

Conductors also shelter internal region from an external $\vec{E}$.



$\vec{E} = 0$ inside empty cavity

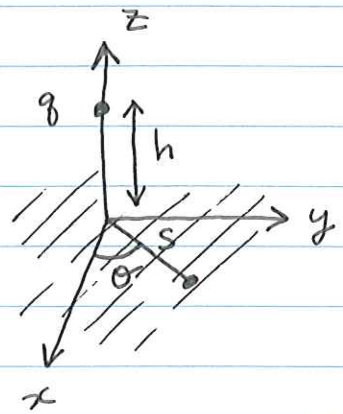


Suppose you have $\vec{E} \neq 0$ inside. Since $\vec{E}$ field lines must begin/end on charges, must go from one surface to another. But $\phi = \text{const}$ on the surface.

Inside a hollow cavity of a conductor, $\vec{E} = 0$
if there is no charge in the cavity.

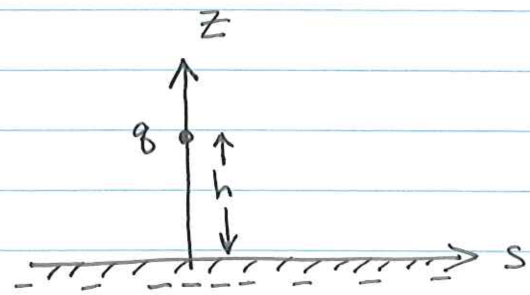
Image charges

Consider a point charge q a distance h above an infinite conducting plane (in the x-y plane).



What is $\vec{E}$ on the surface of the plane?
What is the surface charge density?

Work in cylindrical coords.

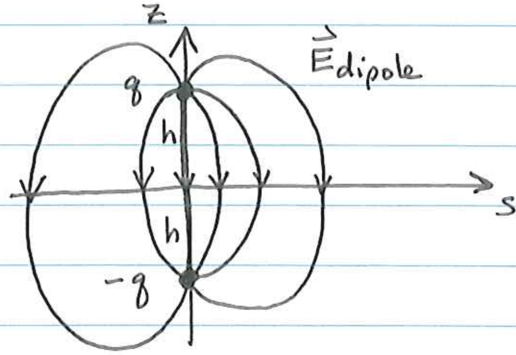


q attracts (−) charges on the conductor (assume $q > 0$)

$$\underbrace{\vec{E}}_{} = \underbrace{\vec{E}_q}_{\text{from } q} + \underbrace{\vec{E}_{plane}}_{\text{from (−) charges on plane.}}$$

Note: $\vec{E}$ must be perpendicular to surface. Must point in $\hat{z}$ direction at the surface ($z=0$)

Method of image charges: borrow a solution from a simpler problem. Dipole: $+q$, $-q$ charges.



Note: $\vec{E}_{dipole}$ is perpendicular to the s-axis at $z=0$.

This $\vec{E}$ is a valid solution for $\vec{E}$ from the original problem.

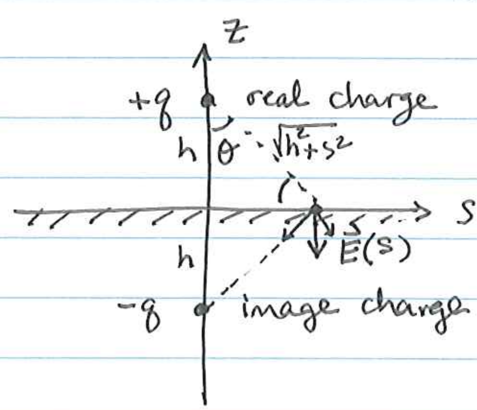
Relies upon mathematical fact called uniquess theorem.

If you can find <u>any</u> valid solution, it is automatically the unique, correct solution.

$$\vec{E}_{dipole} = \vec{E}_q + \vec{E}_{-q} \underset{\text{by uniqueness}}{=} \vec{E}_q + \vec{E}_{plane}$$

$$\Rightarrow \quad \vec{E}_{plane} = \vec{E}_{-q}$$

The effect of charge q on the conducting plane is to make a mirror image charge $-q$.



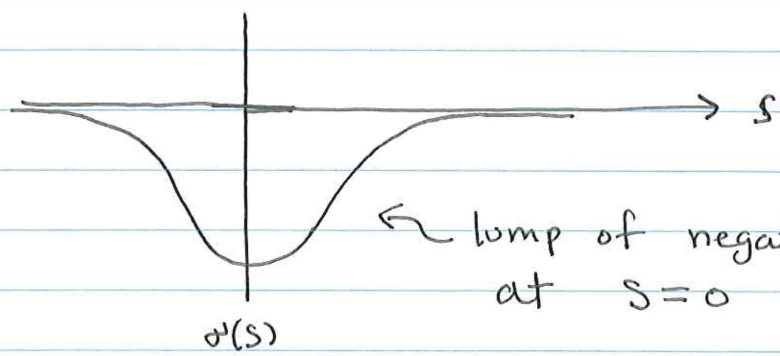
$$\vec{E}(s) = \vec{E}_q + \vec{E}_{-q} \quad \text{at } z=0.$$

$$E_z = -\left(\frac{q}{4\pi\varepsilon_0 (h^2+s^2)} \cos\theta - \frac{q}{4\pi\varepsilon_0 (h^2+s^2)} \cos\theta\right)$$

$$= -\frac{2qh}{4\pi\varepsilon_0 (h^2+s^2)^{3/2}}$$

So $\vec{E}(s) = -\dfrac{2qh}{4\pi\varepsilon_0 (h^2+s^2)^{3/2}} \hat{z}$ on surface $(z=0)$

Surface charge density: $\sigma = \varepsilon_0 E_z = -\dfrac{2qh}{4\pi (h^2+s^2)^{3/2}}$



lump of negative charge centered at $s=0$ underneath charge q.