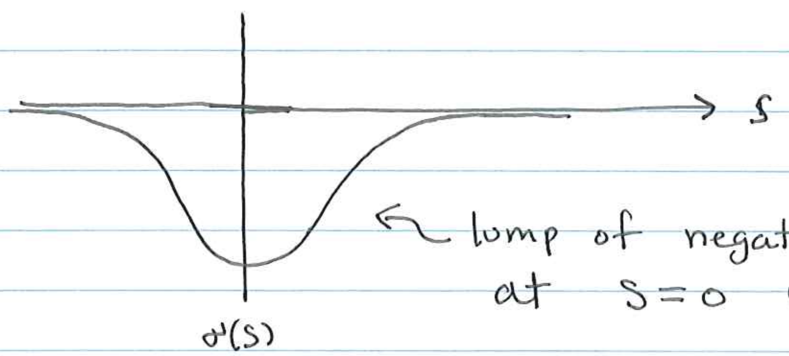
$$\vec{E}(s) = \vec{E}_q + \vec{E}_{-q} \quad \text{at } z=0.$$

$$E_z = -\left(\frac{q}{4\pi\varepsilon_0 (h^2+s^2)} \cos\theta - \frac{q}{4\pi\varepsilon_0 (h^2+s^2)} \cos\theta \right)$$

$$= -\frac{2qh}{4\pi\varepsilon_0 (h^2+s^2)^{3/2}}$$

So $\vec{E}(s) = -\dfrac{2qh}{4\pi\varepsilon_0 (h^2+s^2)^{3/2}} \hat{z}$ on surface $(z=0)$

Surface charge density: $\sigma = \varepsilon_0 E_z = -\dfrac{2qh}{4\pi (h^2+s^2)^{3/2}}$



lump of negative charge centered at $s=0$ underneath charge q.

Capacitors

The charge Q on a conductor is related to the potential on the conductor.

example: spherical conductor with radius R

Q

$$\phi = \frac{Q}{4\pi\varepsilon_0 R} \quad \text{on the conductor}$$

or $Q = 4\pi\varepsilon_0 R\,\phi$

Define constant $C = 4\pi\varepsilon_0 R$, called <u>capacitance</u>.

Then $Q = C\phi$. Note C only depends on the geometry of the conductor (size and orientation) and not on how much charge it has.

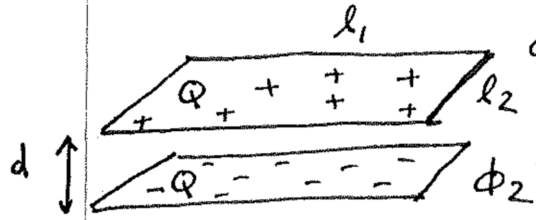
C has units $\frac{\text{Coulomb}}{\text{Volt}} = 1$ farad. Also, C can always be written as $\varepsilon_0 \times$(length scale)

Capacitance can be defined for any number of conductors with charges $Q_1, Q_2, \ldots$ and potentials $\phi_1, \phi_2, \ldots$

The most common example is for two conductors with opposite charges Q and $-Q$, and potentials ϕ_1 and ϕ_2.

<u>example: parallel plate capacitor</u>

Consider two parallel plates of dimensions $l_1 \times l_2$, separated by a distance d, with charges Q & $-Q$ and potentials ϕ_1, ϕ_2

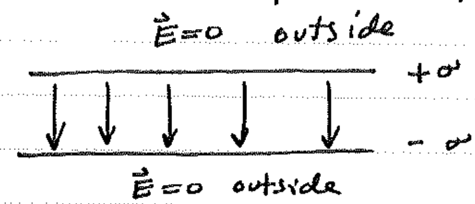


Each plate has area
$$A = l_1 l_2.$$

If Q is uniformly distributed, then surface charge is
$$\sigma = \pm \frac{Q}{A}$$

Approximation: if $d \ll l_1, l_2$ then we can approximate $\vec{E}$ as the same as for infinite parallel plates.

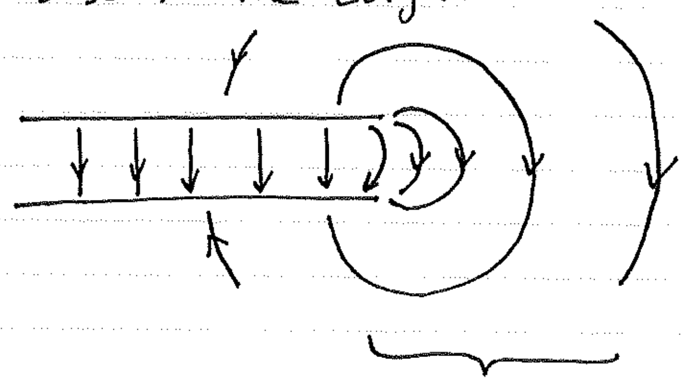
infinite plates:



$E = \frac{\sigma}{\varepsilon_0}$ inside

potential difference: $\phi_1 - \phi_2 = \frac{\sigma d}{\varepsilon_0} = \frac{Qd}{A\varepsilon_0}$

For finite-sized plates, this approximation breaks down close to the edge.



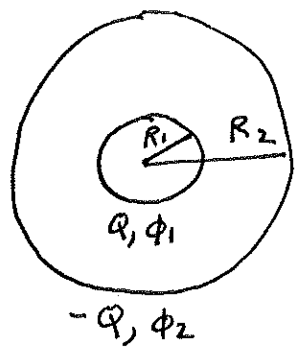
fringe field: only important a distance $\sim d$ from the edge.

Under the infinite plate approximation:

$$\Delta\phi = \frac{d}{A\varepsilon_0} Q \quad \text{or} \quad Q = \underbrace{\frac{A\varepsilon_0}{d}}_{C} \Delta\phi$$

Capacitance $C = \frac{A\varepsilon_0}{d}$

example: spherical capacitor. Consider two concentric spherical conduction shells with radii R_1 & R_2. Let Q and $-Q$ be charges and ϕ_1 and ϕ_2 be the potentials on the shells.



For $R_1 < r < R_2$ (between shells),

$$\vec{E} = \frac{Q}{4\pi\varepsilon_0 r^2}\hat{r}$$

Potential difference:

$$\Delta\phi = \phi_1 - \phi_2 = -\int_{R_2}^{R_1} dr\, \frac{Q}{4\pi\varepsilon_0 r^2}$$

$$= \frac{Q}{4\pi\varepsilon_0 r}\Big|_{R_2}^{R_1} = \frac{Q}{4\pi\varepsilon_0}\left(\frac{1}{R_1} - \frac{1}{R_2}\right)$$

So

$$Q = \underbrace{\frac{4\pi\varepsilon_0}{\frac{1}{R_1} - \frac{1}{R_2}}}_{C}\,\Delta\phi$$

Capacitance $C = \dfrac{4\pi\varepsilon_0}{1/R_1 - 1/R_2}$

Letting $R_2 \to \infty$, we have $C = 4\pi\varepsilon_0 R_1$ as in the first example.

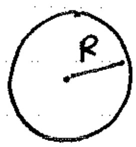
Energy of electric fields

Recall from Ch. 1: the potential energy of a given charge configuration = amount of work required to assemble it.

We did this for point charges. What about for a continuous charge distribution?

Suppose we have a spherical shell of charge q. What is the work required to add some extra charge Δq to the shell?

$\vec{ds} = dr\,\hat{r}$



q

$$\vec{E} = \frac{q}{4\pi\varepsilon_0 r^2}\,\hat{r}$$

$\longleftarrow\bullet\ \Delta q$

$$\Delta W = -\int_{\infty}^{R} dr\ \frac{q}{4\pi\varepsilon_0 r^2}\,\Delta q = \frac{q\,\Delta q}{4\pi\varepsilon_0 R}$$

Next, if we start with a shell with zero charge, what is the total work required to build it up to charge Q by adding Δq charge at a time.

Infinitesimal limit : $\Delta q \to dq$, $\Delta W \to dW$

$$W = \int dW = \int_0^Q \frac{q\,dq}{4\pi\varepsilon_0 R} = \frac{1}{4\pi\varepsilon_0 R}\,\frac{Q^2}{2}$$

Potential energy is $\underline{U = \dfrac{Q^2}{8\pi\varepsilon_0 R}}$

We know that charge density ρ and $\vec{E}$ are equivalent ways to describe a system.

We can also compute U from $\vec{E}$ alone.

$$\boxed{U = \frac{\varepsilon_0}{2} \int_{\substack{\text{all}\\\text{space}}} dV \, |\vec{E}|^2}$$

Potential energy stored in $\vec{E}$ field = Work to assemble charge configuration.

For spherical shell: $\vec{E} = \begin{cases} 0 & r<R \\ \dfrac{Q}{4\pi\varepsilon_0 r^2}\hat{r} & r>R \end{cases}$

Check:

$$U = \frac{\varepsilon_0}{2} \int dV \, \frac{Q^2}{(4\pi\varepsilon_0)^2 r^4}$$

$$= \frac{\varepsilon_0}{2} \int_R^{\infty}\int_0^{\pi}\int_0^{2\pi} r^2 dr \, \sin\theta \, d\theta \, d\phi \quad \frac{Q^2}{(4\pi\varepsilon_0)^2 r^4}$$

$$= \frac{Q^2}{8\pi\varepsilon_0}\frac{1}{4\pi}\underbrace{\int_R^{\infty} dr \, \frac{1}{r^2}}_{1/R} \; \underbrace{\int_0^{\pi} \sin\theta \, d\theta}_{2} \; \underbrace{\int_0^{2\pi} d\phi}_{2\pi}$$

$$= \frac{Q^2}{8\pi\varepsilon_0 R}$$

Proof: First, recall integration by parts.
Suppose you have two functions $f(x)$ and $g(x)$, which both go to zero for $x \to \pm\infty$.

Then integration by parts tells us that

$$\int_{-\infty}^{\infty} dx\, f(x)\, g'(x) = f(x)\, g(x) \Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} dx\, f'(x)\, g(x)$$

$$= - \int_{-\infty}^{\infty} dx\, f'(x)\, g(x)$$

The potential energy of a point charge configuration is

$$U = \frac{1}{2} \sum_{\substack{i,j \\ i \neq j}} \frac{q_i q_j}{4\pi\varepsilon_0 r_{ij}} = \frac{1}{2} \sum_i q_i \phi_i$$

where $\phi_i = \sum_{\substack{j \\ j \neq i}} \frac{q_j}{4\pi\varepsilon_0 r_{ij}}$ is the potential that q_i feels due to all the other point charges j $(j \neq i)$.

For a continuous distribution: $q_i \to dq = \rho(\vec{r})\, dV$

$\Sigma_i \to \int_{\text{all space}}$

$\phi_i \to \phi(\vec{r})$

So $$U = \frac{1}{2} \int_{\substack{\text{all} \\ \text{space}}} dV\, \rho(\vec{r})\, \phi(\vec{r})$$

Note: $\int_{\substack{\text{all} \\ \text{space}}} dV = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} dz$

Use Gauss's law (differential form): $\rho = \varepsilon_0 \vec{\nabla} \cdot \vec{E}$

$$U = \frac{\varepsilon_0}{2} \int dV \left(\frac{\partial E_x}{\partial x} \phi + \frac{\partial E_y}{\partial y} \phi + \frac{\partial E_z}{\partial z} \phi \right)$$

$$= -\frac{\varepsilon_0}{2} \int dV \left(E_x \frac{\partial E_x}{\partial x} + E_y \frac{\partial \phi}{\partial y} + E_z \frac{\partial \phi}{\partial z} \right)$$ int. by parts

$$= -\frac{\varepsilon_0}{2} \int dV \, \vec{E} \cdot \vec{\nabla} \phi = \frac{\varepsilon_0}{2} \int dV \, |\vec{E}|^2$$ since $\vec{E} = -\vec{\nabla}\phi$.

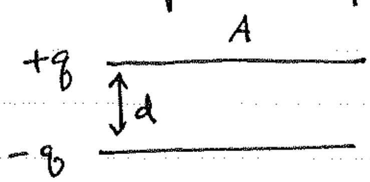
Energy Stored in a capacitor

For a spherical conducting sphere*, we found

* since Q is distributed in a shell

$$U = \frac{Q^2}{8\pi\varepsilon_0 R} = \frac{1}{2} \frac{Q^2}{C}, \quad C = 4\pi\varepsilon_0 R.$$

Parallel plate capacitor:



Work required to move charge dq from the − plate to positive plate is:

$$dW = dq \Delta\phi = \frac{dq \, q}{C}$$

Therefore the total potential energy to have charges $+Q$ and $-Q$ on the plates is:

$$U = \int dW = \int_0^Q \frac{q \, dq}{C} = \frac{Q^2}{2C}$$