V. Special Relativity & Magnetism

For a charge q at rest, $\vec{F} = q\vec{E}$.

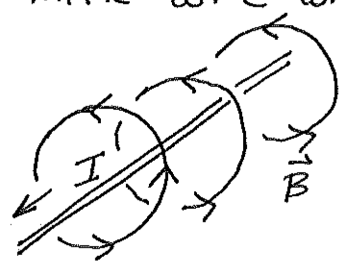
For a charge moving with velocity $\vec{v}$, there is another velocity-dependent contribution to the force:

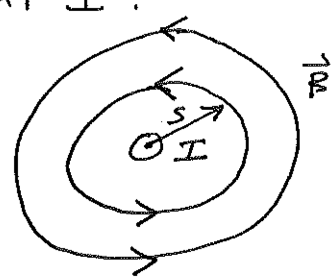
$$\boxed{\vec{F} = q\vec{E} + q\vec{v} \times \vec{B}} \quad \text{Lorentz force law}$$

where $\vec{B}$ is the magnetic field.

$\vec{E}$ is caused by charges, $\vec{B}$ is caused by currents.

e.g. infinite wire with current I.





I flowing out of page.

magnitude of $\vec{B}$ is $\boxed{B = \frac{\mu_0 I}{2\pi s}}$

Why do we have $\vec{E}$ and $\vec{B}$ fields?
Historically, electricity & magnetism were separate (but related) phenomena. In 1905, Einstein showed that $\vec{E}$ and $\vec{B}$ were manifestations of the <u>same</u> phenomenon as a consequence of special relativity.

$\vec{E}$ fields in one reference frame are $\vec{B}$ fields in another reference frame.

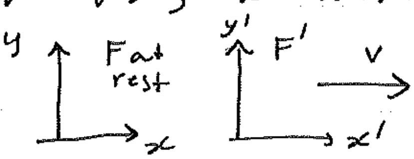
In fact, it was electricity & magnetism that inspired Einstein to discover special relativity: his 1905 special relativity paper was titled "On the electrodynamics of moving bodies"

Here our goal is to show that electric forces + special relativity implies the necessity of magnetic forces.

Review of special relativity

Any "event" has a time t and position (x, y, z). In one reference frame (F), the time between events is Δt and the separation is $(\Delta x, \Delta y, \Delta z)$.

But in a different reference frame F', moving with respect to F by velocity $\vec{V} = v\hat{x}$, we have



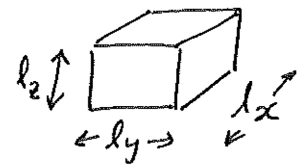
$$\left.\begin{aligned} \Delta t' &= \gamma \Delta t - \beta\gamma \Delta x / c \\ \Delta x' &= \gamma \Delta x - \beta \gamma c \Delta t \\ \Delta y' &= \Delta y \\ \Delta z' &= \Delta z \end{aligned}\right\} \text{Lorentz transformations}$$

where $\beta = v/c$, $\gamma = \frac{1}{\sqrt{1-\beta^2}}$, $c =$ speed of light.

This has some interesting consequences:

Note: $\beta \leq 1$
$\gamma \geq 1$

(i) Length contraction.



Suppose we have an object at rest in F' that has length (l_x, l_y, l_z) in the x, y, z-directions. i.e. $\Delta x' = l_x$, $\Delta y' = l_y$, $\Delta z' = l_z$.

How big is it in frame F where it is moving at velocity v? We have:

$$\Delta x' = l_x = \Delta x\, \gamma$$
$$\Delta y' = l_y = \Delta y$$
$$\Delta z' = l_z = \Delta z$$

(we assume measuring its sides $\Delta x, \Delta y, \Delta z$ ~~at~~ all at the same time so $\Delta t = 0$)

$$\Rightarrow \underbrace{\Delta x = l_x/\gamma}_{\text{distance of side along velocity direction shrinks by } 1/\gamma}, \quad \Delta y = l_y, \quad \Delta z = l_z$$

(ii) Time dilation.

Suppose we have a clock at rest in frame F' that ticks ~~[illegible]~~ once every ~~[illegible]~~ time τ.

$$\Rightarrow \Delta t' = \tau, \quad \Delta x' = \Delta y' = \Delta z' = 0.$$

How often does it tick in frame F?

$$\tau = \Delta t' = \gamma \Delta t - \beta \gamma \Delta x / c$$

$$0 = \Delta x' = \gamma \Delta x - \beta \gamma c \, \Delta t \quad \longrightarrow \quad \Delta x = \beta c \, \Delta t$$

$$\tau = \gamma (1 - \beta^2) \Delta t = \frac{1}{\gamma} \Delta t$$

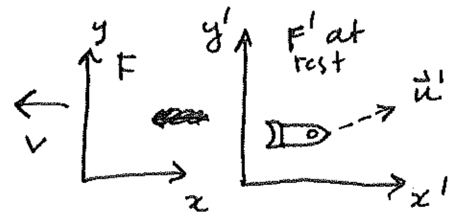
$$\Rightarrow \boxed{\Delta t = \gamma \tau}$$

Ticks in frame F are lengthened by a factor of γ.

To an observer at rest (F), a moving clock runs slower and a moving ruler is shorter (along direction of motion)

(iii) Addition of velocities

A spaceship at rest in F' launches a bullet with velocity $\vec{u}' = (u_x', u_y', u_z')$. What is the bullet velocity $\vec{u}$ to an observer at rest in F?



Invert Lorentz transformation rules:

$$\Delta t = \gamma \Delta t' + \beta \gamma \Delta x' / c$$
$$\Delta x = \gamma \Delta x' + \beta \gamma c \Delta t'$$
$$\Delta y = \Delta y'$$
$$\Delta z = \Delta z'$$

Bullet velocity in F' is:

$$\vec{u}' = (u'_x, u'_y, u'_z) = \left(\frac{\Delta x'}{\Delta t'}, \frac{\Delta y'}{\Delta t'}, \frac{\Delta z'}{\Delta t'} \right)$$

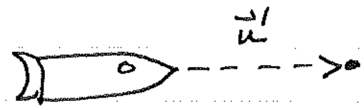
Bullet velocity in F is:

$$\vec{u} = (u_x, u_y, u_z) = \left(\frac{\Delta x}{\Delta t}, \frac{\Delta y}{\Delta t}, \frac{\Delta z}{\Delta t} \right)$$

Use Lorentz transformations to relate $\vec{u}$ and $\vec{u}'$.

<u>Simple case</u>: bullet in same direction as $\vec{v}$ (x-direction)

$$\vec{u}' = (u'_x, 0, 0)$$



Then $$u_x = \frac{\Delta x}{\Delta t} = \frac{\gamma \Delta x' + \beta \gamma c \Delta t'}{\gamma \Delta t' + \beta \gamma \Delta x'/c}$$

$$= \frac{\left(\frac{\Delta x'}{\Delta t'}\right) + \beta c}{1 + \beta \left(\frac{\Delta x'}{\Delta t'}\right)/c} = \frac{u'_x + \beta c}{1 + \beta u'_x/c}$$

Let $\beta_u = u_x/c$ and $\beta'_u = u'_x/c$, then

$$\boxed{\beta_u = \frac{\beta'_u + \beta}{1 + \beta \beta'_u}}$$

And we also have the inverse relation:

$$\boxed{\beta'_u = \frac{\beta_u - \beta}{1 - \beta \beta_u}}$$

flipping $\vec{u} \leftrightarrow \vec{u}'$
means $\beta \rightarrow -\beta$.

example: spaceship moving at $v = 0.9c$ fires bullet at $u' = 0.9c$. What is bullet velocity u to observer at rest in F?

$$\beta_u = \frac{0.9 + 0.9}{1 + (0.9)(0.9)} = \frac{1.8}{1.81} \approx 0.994$$

$$u = 0.994c$$

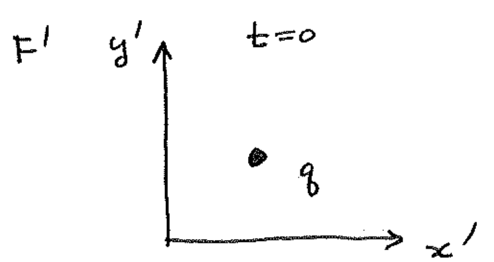
example: now consider a laser, instead of a bullet, fired at $u' = c$. What is the laser velocity u in F?

$$\beta_u = \frac{0.9 + 1}{1 + (0.9)(1)} = \frac{1.9}{1.9} = 1$$

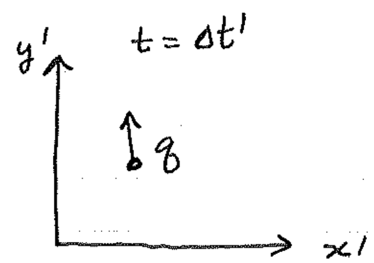
$$u = c$$

Speed of light is the same to all observers.

(iv). <u>Forces</u>. Suppose a charge q is initially at rest in frame F' and then it experiences a force $\vec{F}'$ over a small interval $\Delta t'$.



at rest



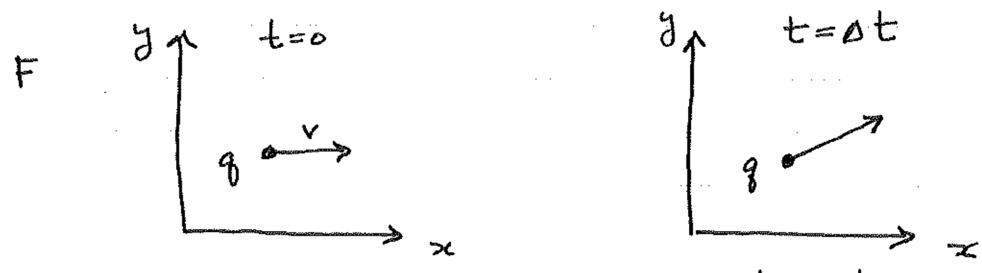
gets a kick in y direction.

$$\vec{F}' = F'_y \, \hat{y}$$

The change in momentum in the y direction is

$$\Delta p_y' = F_y' \, \Delta t' \qquad \text{or} \qquad F_y' = \frac{\Delta p_y'}{\Delta t'}$$

Now consider frame F where q is initially moving with velocity $\vec{v} = v\hat{x}$.



moving in x direction.

gets a kick in y direction.

Since going from F' to F only affects motion in x direction, momentum in y-direction is conserved.

$$\Delta p_y = \Delta p_y'$$

However the time interval is changed: $\Delta t = \gamma \Delta t'$
The force acts over a longer time by time dilation.
So the force in frame F is measured to be

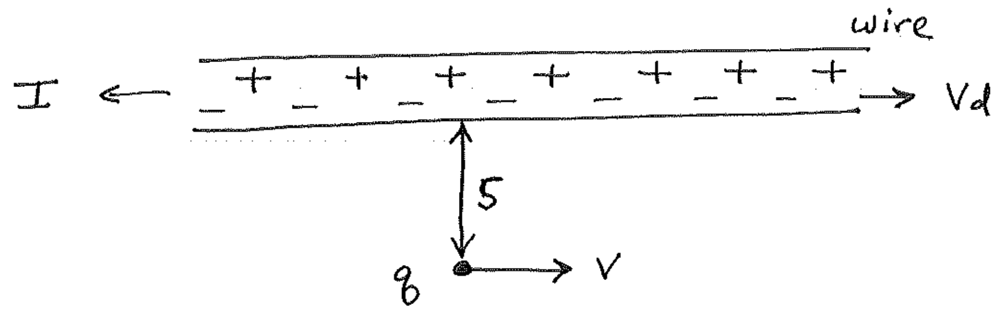
$$F_y = \frac{\Delta p_y}{\Delta t} = \frac{\Delta p_y'}{\gamma \Delta t'} = F_y' / \gamma$$

The perpendicular force acting on a moving particle q is reduced by a factor $1/\gamma$ relative to what the particle feels in its rest frame.

(Note: this is only for a perpendicular force. A parallel force in the x-direction would be $F_x = F_x'$.)

Magnetic force: Combine all these relativistic effects to derive magnetic force between moving charge q and wire with current I.

Lab frame F:



Point charge q is distance s from wire, moving parallel with velocity v.

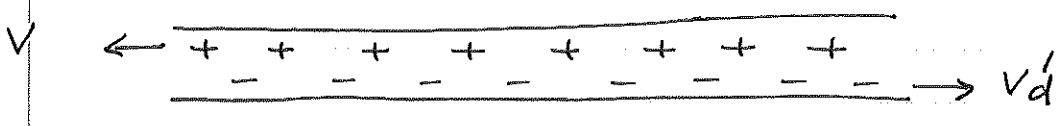
Wire has fixed positive charges (ions) while negative charges (electrons) go right with drift velocity V_d.

$$\left.\begin{array}{l} \lambda = \text{charge density in ions} \\ -\lambda = \text{charge density in electrons} \end{array}\right\} \text{wire is neutral.}$$

$$\text{current } I = \frac{\text{charge}}{\text{time}} = \frac{\text{charge}}{\text{length}} \times \frac{\text{length}}{\text{time}} = \lambda V_d$$

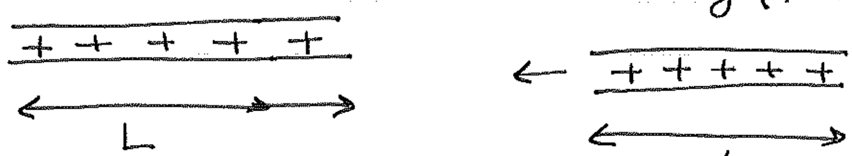
In frame F, there is no electric field acting on q since wire is neutral.

Frame F', rest frame of q:



Now ions are moving left with velocity V.
Consider a length L containing charge Q in ions.

At rest (F) moving (F')



$L' = L/\gamma$
by length contraction.

Charge density at rest (F): $\lambda = \dfrac{Q}{L}$

Charge density in F': $\dfrac{Q}{L'} = \gamma \dfrac{Q}{L} = \gamma \lambda$

New drift velocity for electrons is v_d'. Use rules for adding velocities:

Define $\beta_d' = v_d'/c$, $\beta_d = v_d/c$, $\beta = V/c$

Then
$$\beta_d' = \frac{\beta_d - \beta}{1 - \beta\beta_d}$$

Consider a length L containing $-Q$ charge in electrons:

What is L' in terms of L? Let L_0 be length of L in electrons' rest frame.

$$L = \frac{L_0}{\gamma_d} \qquad \gamma_d = \frac{1}{\sqrt{1-\beta_d^2}}$$

$$L' = \frac{L_0}{\gamma_d'} \qquad \gamma_d' = \frac{1}{\sqrt{1-\beta_d'^2}}$$

$$\Rightarrow \quad L' = \frac{\gamma_d}{\gamma_d'} L = \sqrt{\frac{1-\beta_d'^2}{1-\beta_d^2}}\; L$$

$$1-\beta_d'^2 = 1 - \left(\frac{\beta_d - \beta}{1-\beta\beta_d}\right)^2 = \frac{(1-\beta\beta_d)^2 - (\beta_d-\beta)^2}{(1-\beta\beta_d)^2}$$

$$= \frac{1 - 2\beta\beta_d + \beta^2\beta_d^2 - \beta_d^2 - \beta^2 + \cancel{2\beta_d\beta}}{(1-\beta\beta_d)^2}$$

$$= \frac{(1-\beta_d^2)(1-\beta^2)}{(1-\beta\beta_d)^2} = \frac{1}{\gamma^2\gamma_d^2(1-\beta\beta_d)^2}$$

So we have
$$L' = \frac{\gamma_d}{\gamma\gamma_d(1-\beta\beta_d)} L$$
$$= \frac{1}{\gamma(1-\beta\beta_d)} L$$

Electron charge density is in F':
$$\frac{-Q}{L'} = \frac{-Q}{L}\gamma(1-\beta\beta_d) = -\lambda\gamma(1-\beta\beta_d)$$

In rest frame of q:

Ions have charge density $= \lambda\gamma$
electrons have charge density $= -\lambda\gamma(1-\beta\beta_d)$

They are not equal! The wire has a net positive charge density:

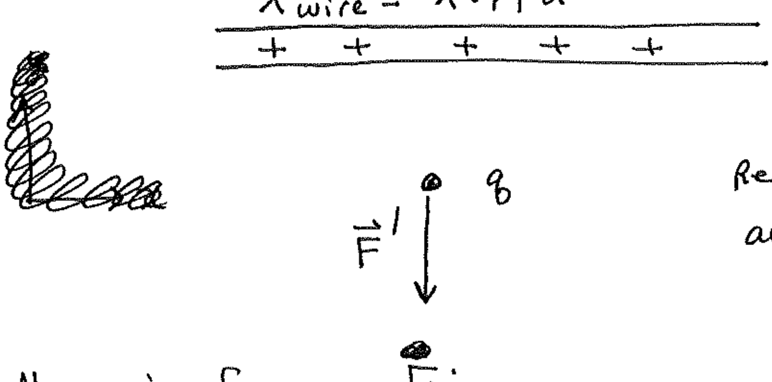
$$\lambda\gamma - \lambda\gamma(1-\beta\beta_d) = \lambda\gamma\beta\beta_d$$

The wire is positively charged and generates an $\vec{E}$ field:

$$E' = \frac{\lambda\gamma\beta\beta_d}{2\pi\varepsilon_0 S}$$

q feels a force $\quad F' = qE' = \dfrac{q\lambda\gamma\beta\beta_d}{2\pi\varepsilon_0 S}$

$\lambda_{wire} = \lambda\gamma\beta\beta_d$



Repulsive force points away from the wire.

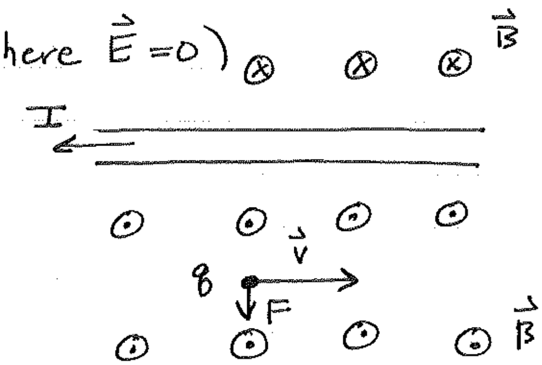
Now in frame F:

$$F = \frac{F'}{\gamma} = \frac{q\lambda\beta\beta_d}{2\pi\varepsilon_0 S} = \frac{q\lambda v v_d}{2\pi\varepsilon_0 c^2 S}$$

$$= q v \underbrace{\frac{I}{2\pi\varepsilon_0 c^2 S}}_{B}$$

Lorentz force law: (where $\vec{E}=0$)

$$\vec{F} = q\,\vec{v} \times \vec{B}$$



$\vec{F}$ points away from the wire with magnitude

$$F = q\,v\,B = q\,v\,\frac{\mu_0 I}{2\pi s}$$

We get the same result using special relativity. Moreover: $\mu_0 = \frac{1}{\varepsilon_0 c^2}$

So μ_0, ε_0 are not independent constants, but

$$\boxed{\mu_0 \varepsilon_0 = \frac{1}{c^2}}$$