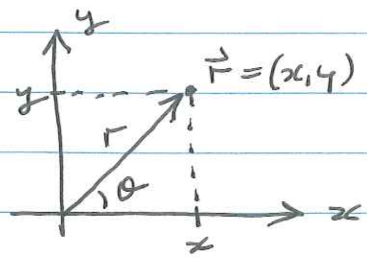
Tutorial #1: Curvilinear coordinate systems

In 2D, we have $\vec{r} = (x, y)$ (Cartesian)
Can also use polar coords (r, θ)

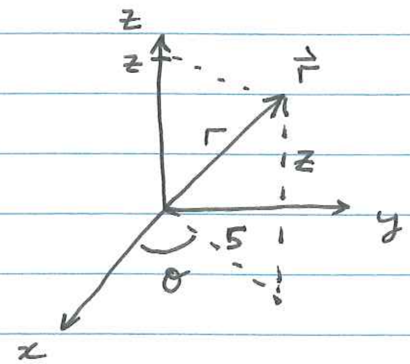


$$r = |\vec{r}| = \sqrt{x^2 + y^2}$$

$$\left.\begin{array}{l} x = r\cos\theta \\ y = r\sin\theta \end{array}\right\} \Rightarrow \begin{array}{l} \tan\theta = \frac{x}{y} \\ \theta = \arctan\left(x/y\right) \end{array}$$

In 3D, we have $\vec{r} = (x, y, z)$ (Cartesian)

Cylindrical coords: (s, θ, z)



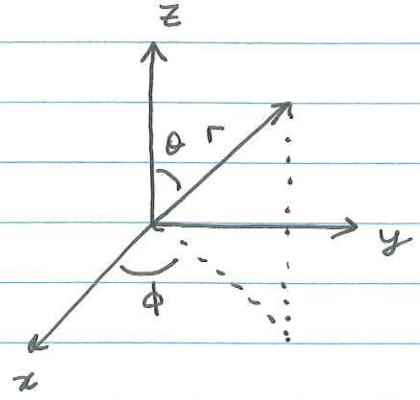
$$s = \sqrt{x^2 + y^2}$$

$$z = z$$

$$\left.\begin{array}{l} x = s\cos\theta \\ y = s\sin\theta \end{array}\right\} \Rightarrow \begin{array}{l} \tan\theta = \frac{x}{y} \\ \theta = \arctan x/y \end{array}$$

$$r = |\vec{r}| = \sqrt{x^2 + y^2 + z^2} = \sqrt{s^2 + z^2}$$

Spherical coords: (r, θ, ϕ)



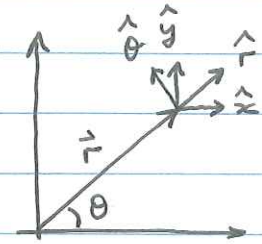
$$r = \sqrt{x^2 + y^2 + z^2}$$

$$z = r\cos\theta$$

$$x = r\sin\theta\cos\phi$$

$$y = r\sin\theta\sin\phi$$

Unit vectors in curvilinear systems:



2D: Cartesian: $\hat{x} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\hat{y} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

Polar: $\hat{r} = \vec{r}/r = \frac{1}{r}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$

$\hat{\theta} =$ ~~direction from moving in~~ oriented along the "θ-direction" perpendicular to $\hat{r}$

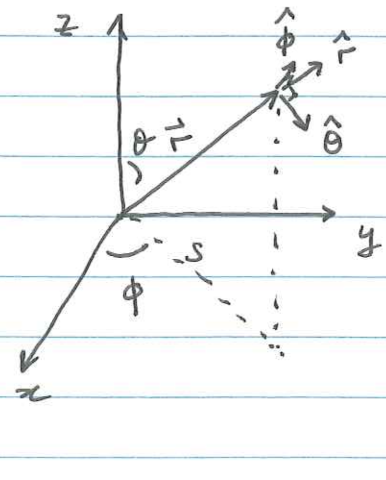
$$= \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}$$

3D: Cartesian: $\hat{x} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $\hat{y} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, $\hat{z} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

Cylindrical: $\hat{s} = \begin{pmatrix} \cos\theta \\ \sin\theta \\ 0 \end{pmatrix}$, $\hat{z} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$, $\hat{\theta} = \begin{pmatrix} -\sin\theta \\ \cos\theta \\ 0 \end{pmatrix}$

Spherical: $\hat{r} = \frac{\vec{r}}{r} = \begin{pmatrix} \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos\theta \end{pmatrix}$, also $\hat{\theta}, \hat{\phi}$ as well.

$$= \frac{1}{r}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$



$s = r\sin\theta$

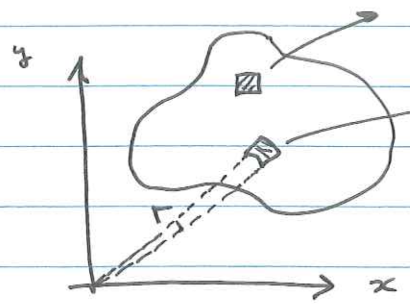
$z = r\cos\theta$

$x = s\cos\phi = r\sin\theta\cos\phi$

$y = s\sin\phi = r\sin\theta\sin\phi$

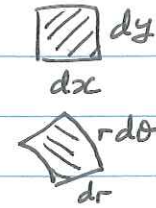
Calculus in curvilinear coordinates:

Compute areas & volumes



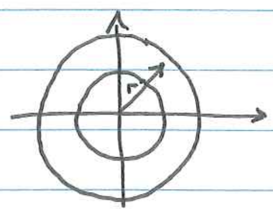
$dA = dx\,dy$ Cartesian

$dA = r\,dr\,d\theta$ Polar



e.g. rectangle ~~square~~ $A = \int_0^h dy \int_0^b dx = hb$

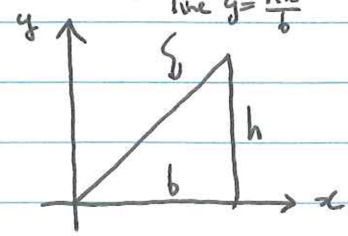
ex. compute area of annulus of inner radius R_1 and outer radius R_2



$$A = \int dA = \iint r\,dr\,d\theta = \int_{R_1}^{R_2} dr\, r \underbrace{\int_0^{2\pi} d\theta}_{\text{circumference of unit circle.}}$$

$$= \frac{1}{2}\left(R_2^2 - R_1^2\right) 2\pi = \pi\left(R_2^2 - R_1^2\right)$$

ex. compute area of triangle with base length b and height h.

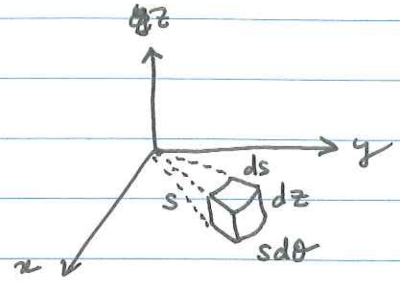


$$A = \int dA = \iint dx\,dy = \int_0^b dx \int_0^{\frac{hx}{b}} dy$$

Note: at each value x, only integrate up to $y = \frac{hx}{b}$

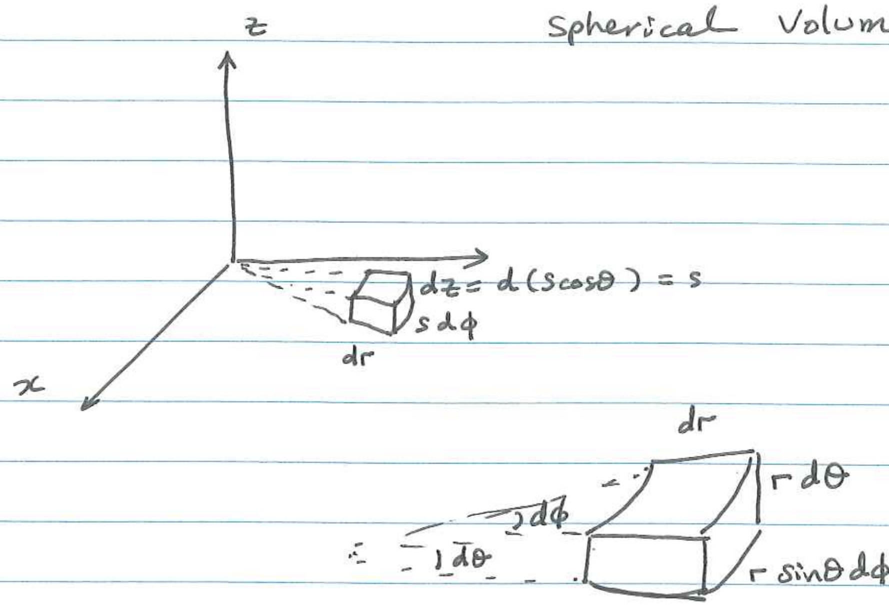
$$A = \int_0^b dx\left(\frac{hx}{b}\right) = \frac{1}{2}\frac{h}{b}x^2\Big|_0^b = \frac{1}{2}hb$$

cylindrical volume element:

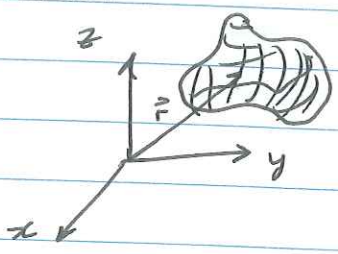


$dV = s\,d\theta \cdot dz\,ds = s\,ds\,d\theta\,dz$

spherical volume element:



Volumes:



$dV = dx\,dy\,dz$

Cartesian

$dV = s\,ds\,d\theta\,dz$ Cylindrical

$dV = r^2 \sin\theta\,dr\,d\theta\,d\phi$ Spherical

ex. volume of a sphere of radius R:

$$V = \int dV = \iiint r^2 \sin\theta\,dr\,d\theta\,d\phi = \int_0^R r^2 dr \int_0^\pi d\theta \sin\theta \int_0^{2\pi} d\phi$$

$$= \frac{1}{3} R^3 \cdot 2 \cdot 2\pi = \frac{4\pi}{3} R^3$$