# PHYS 3090: Homework 1 (due Friday Sept. 19)

1. This problem deals with complex vector spaces.

   - Let $\vec{u} = \begin{pmatrix} 2i \\ -1 \end{pmatrix}$, $\vec{v} = \begin{pmatrix} 1+i \\ -i \end{pmatrix}$, and $a = 3i + 1$. Compute $\langle \vec{u}, \vec{v} \rangle$ and $\langle a\vec{u}, \vec{v} \rangle$.
   - Let $\vec{u}, \vec{v}$ be vectors in a complex vector space and $R$ be a unitary matrix. Show that $\langle \vec{u}, \vec{v} \rangle = \langle \vec{u}', \vec{v}' \rangle$, where $\vec{u}' = R\vec{u}$ and $\vec{v}' = R\vec{v}$.
   - Prove that the eigenvalues of a Hermitian matrix $M$ are real.
   - Find the eigenvalues $\lambda_1, \lambda_2$ for the Hermitian matrix $M = \begin{pmatrix} 0 & 2i \\ -2i & 3 \end{pmatrix}$. Find the matrix $R$ such that $R^\dagger M R = \text{diag}(\lambda_1, \lambda_2)$.

2. In the quantum mechanical description of angular momentum, the **Pauli matrices**

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{1}$$

play an important role. Defining $\vec{\boldsymbol{\sigma}} = \begin{pmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \end{pmatrix}$, any real three-component vector $\vec{v} = \begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix}$ can be expressed as a $2 \times 2$ Hermitian matrix

$$\vec{v} \cdot \vec{\boldsymbol{\sigma}} = \sum_{i=1}^{3} v_i \sigma_i = \begin{pmatrix} v_z & v_x - iv_y \\ v_x + iv_y & -v_z \end{pmatrix} \tag{2}$$

   - Show that $(\vec{v} \cdot \vec{\boldsymbol{\sigma}})^2 = |\vec{v}|^2 \, \mathbb{1}$ for any $\vec{v}$.
   - Show that $\exp(i\vec{v} \cdot \vec{\boldsymbol{\sigma}}) = \cos|\vec{v}| \, \mathbb{1} + i(\hat{v} \cdot \vec{\boldsymbol{\sigma}}) \sin|\vec{v}|$.
   - Consider a new matrix $M = c\mathbb{1} + \vec{v} \cdot \vec{\boldsymbol{\sigma}}$. What is $\exp(iM)$? Compute $\exp \begin{pmatrix} 2i & 1 \\ -1 & 0 \end{pmatrix}$.

3. Consider a vector $\vec{v} = \begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix}$ in $\mathbb{R}^3$ (3d space).

   - Find the $3 \times 3$ rotation matrix $R_z(\alpha)$ that rotates $\vec{v}$ about the $\hat{e}_z$ axis by an angle $\alpha$. Similarly, find the rotation matrices $R_x(\alpha)$ and $R_y(\alpha)$ that rotate $\vec{v}$ about the $\hat{e}_x$ and $\hat{e}_y$ axes, respectively.
   - Show that two rotations about the same axis commute by proving $R_z(\alpha)R_z(\theta) = R_z(\theta)R_z(\alpha)$. Show that two rotations about different axes do not commute. (Note: Two matrices $A$ and $B$ **commute** if $AB = BA$.)

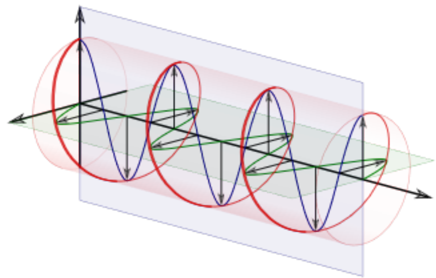

Figure 1: Sketch of a circularly polarized state rotating in the $x$-$y$ plane as a function of time. The blue curve is $x(t)$, the green curve is $y(t)$, and the red curve is $\vec{r}(t)$.

4. The equation of motion for a 3d simple harmonic oscillator is $\ddot{\vec{r}} = -\omega^2 \vec{r}$, where $\vec{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ is the displacement vector, $\omega$ is the frequency, and $\dot{} = d/dt$.

- Show that $\vec{r}(t)$ given by the real part of the complex vector $\vec{\mathscr{R}}(t) = \vec{v} e^{i\omega t}$ is a solution for any vector $\vec{v}$.
- What are the eigenvalues $(\lambda_1, \lambda_2, \lambda_3)$ and eigenvectors $(\vec{u}_1, \vec{u}_2, \vec{u}_3)$ of the rotation matrix $R_z(\alpha)$ from problem 3?
- Evaluate $\vec{r}(t)$ separately for each eigenvector by setting $\vec{v} = \vec{u}_i$. Normalize your three solutions such that $|\vec{r}(0)| = r_0$.
- Evaluate the angular momentum vector $\vec{L} = \vec{r} \times \vec{p}$ for each of the solutions obtained above (recall $\vec{p} = m\dot{\vec{r}}$, where $m$ is the mass). Show that the solutions correspond to a right-handed circular polarization ($L_z = |\vec{L}|$), a left-handed circular polarization ($L_z = -|\vec{L}|$), and a longitudinal polarization ($L_z = 0$). (See Figure 1.)
- Evaluate $R_z(\alpha)\vec{r}(t)$ for each solution $\vec{r}(t)$. What is the physical interpretation for your result?