PHYS 3090: Homework 6 (due Tuesday Nov. 4)

Problem 1: Evaluate the c_{-1} term in the Laurent expansion for the following functions

- $f(z) = \frac{\cot z}{z^2}$ about $z = 0$
- $f(z) = \frac{e^z}{z^2+1}$ about $z = i$
- $f(z) = \cos(z + \frac{1}{z})$ about $z = 0$

Problem 2: Compute $\int_0^{2\pi} d\theta \, \frac{1}{(2-\cos\theta)^2}$ using contour integration.

Problem 3: Compute $\oint_C dz \, e^{a/z}$ where C is the unit circle $|z| = 1$ and a is a complex number.

Problem 4: Compute the principal value of $\int_{-\infty}^{\infty} dx \, \frac{\cos x - 1}{x^2}$ by contour integration.

Problem 5 (Bonus): Compute $\oint_C dz \, \frac{e^{1/z}}{1-z}$ where C is the circle $|z| = 0.1$.

Hint: Recall the infinite geometric series formula $\sum_{n=1}^{\infty} x^n = \frac{1}{1-x}$ for $|x| < 1$.

Problem 6 (Bonus): Compute $\int_0^{\infty} dx \, \frac{1}{1+x^{100}}$ by contour integration.

Hint 1: You may use the result from problem 3 in HW 5 to compute the residues.

Hint 2: Your intermediate steps may involve a finite geometric series, which can be summed as follows

$$\sum_{n=0}^{N} x^n = \frac{1 - x^{N+1}}{1-x} \,. \tag{1}$$

Problem 7: Compute the integral $I = \int_{-\infty}^{\infty} dx \, \frac{e^x}{e^{3x}+1}$. We are going to evaluate I using a rectangular contour C shown in Fig. 1, with sides labeled by paths P_1 - P_4.

- Show that $f(z) = \frac{e^z}{e^{3z}+1}$ has poles along the imaginary axis at $z = \frac{\pi i}{3} + \frac{2\pi i n}{3}$, where $n = 0, \pm 1$, *etc.*
- The original integral is $I = \lim_{R\to\infty} \int_{P_1} dz \, f(z)$. Show that $\lim_{R\to\infty} \int_{P_3} dz \, f(z) = -e^{2\pi i/3} \, I$.

  Hint: Use $z = x + \frac{2\pi i}{3}$ along P_3.
- Show that $\lim_{R\to\infty} \int_{P_2} dz \, f(z) = \lim_{R\to\infty} \int_{P_4} dz \, f(z) = 0$.

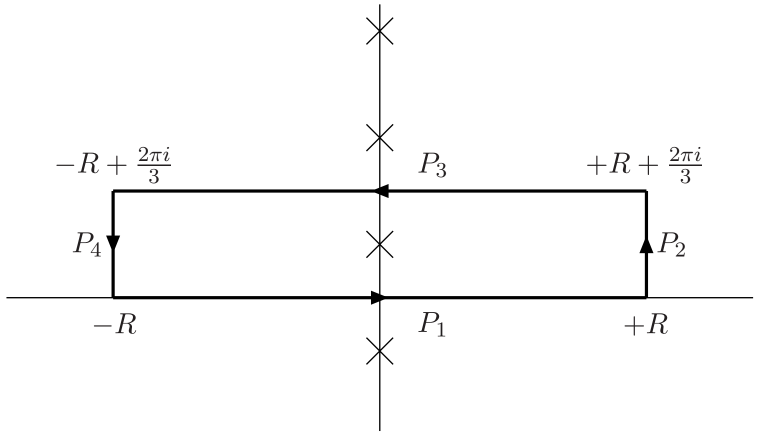

Figure 1: Rectangular contour C defined by the corners $z = \pm R$ and $z = \pm R + \frac{2\pi i}{3}$, and sides P_1-P_4. The crosses show the poles.

- Combining these results, you have

$$\lim_{R\to\infty} \int_{P_1} dz\, f(z) + \int_{P_2} dz\, f(z) + \int_{P_3} dz\, f(z) + \int_{P_4} dz\, f(z) = \left(1 - e^{2\pi i/3}\right) I = \oint_C dz\, f(z)\,. \tag{2}$$

  Now use the residue theorem to evaluate the right-hand side and solve for the original integral I.

Note: This problem is similar to Example 2.22 in the textbook.