I. Topics in Linear Algebra

Vectors & matrices

<u>Vectors</u> $\vec{u}, \vec{v}, \vec{w}, \ldots$ are the set of objects making up a <u>vector space</u> V with the following properties:

(1) Closure under addition: for any $\vec{u}, \vec{v}$ in V, $\vec{w} = \vec{u} + \vec{v}$ is also in V.

(2) Commutivity under addition: $\vec{u} + \vec{v} = \vec{v} + \vec{u}$

(3) Associativity under addition: $(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$

(4) Identity under addition: there exists a zero vector $\vec{0}$ such that $\vec{u} + \vec{0} = \vec{u}$

(5) Inverse under addition: for every vector $\vec{u}$ in V, there exists an inverse vector $(-\vec{u})$ such that $\vec{u} + (-\vec{u}) = \vec{0}$.

Let $a, b, \ldots$ be scalar numbers (real or complex numbers).

(6) Closure under multiplication: for any scalar a and vector $\vec{u}$ in V, $\vec{v} = a\vec{u}$ is also in V.

(7) Associativity under multiplication: $a(b\vec{u}) = (ab)\vec{u}$

(8) Identity under multiplication: $1\vec{u} = \vec{u}$

(9) Distributive properties:

$$a(\vec{u}+\vec{v}) = a\vec{u} + a\vec{v}$$

$$(a+b)\vec{u} = a\vec{u} + b\vec{u}$$

Vector spaces can be real or complex, depending on whether scalars $a, b, \ldots$ are allowed to be real or complex.

Basis vectors are a set of N vectors $\vec{e}_1, \ldots, \vec{e}_N$ in V such that any vector $\vec{v}$ in V can be written as

$$\vec{v} = \sum_{i=1}^{N} v_i \vec{e}_i$$

represent $\vec{v}$ by an $N \times 1$ matrix $\vec{v} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_N \end{pmatrix}$

v_i are the components of $\vec{v}$, which depends on the basis.

Inner product (also known as the dot product) is a way of taking two vectors and turning them into a scalar.

dot product : $\vec{u} \cdot \vec{v} =$ ~~real or comp~~ a number

use alternate notation: $\langle \vec{u}, \vec{v} \rangle$ = real or complex number for real or complex vector space

Inner product should satisfy the following rules:

(1) Additivity: $\langle \vec{u}+\vec{v}, \vec{w} \rangle = \langle \vec{u}, \vec{w} \rangle + \langle \vec{v}, \vec{w} \rangle$

(2) Linear under multiplication: $\langle a\vec{u}, \vec{v} \rangle = a\langle \vec{u}, \vec{v} \rangle$

(3) Positive-definiteness: $\langle \vec{u}, \vec{u} \rangle \geq 0$

and $\langle \vec{u}, \vec{u} \rangle = 0$ only for $\vec{u} = \vec{0}$ (zero vector)

(4) Conjugation: $\langle \vec{u}, \vec{v} \rangle = \begin{cases} \langle \vec{v}, \vec{u} \rangle & \text{real vector space} \\ \langle \vec{v}, \vec{u} \rangle^* & \text{complex vector space} \end{cases}$

Norm of vector is $|\vec{v}| = \sqrt{\langle \vec{v}, \vec{v} \rangle}$.

Note: for a complex vector space

$$\langle a\vec{u}, \vec{v} \rangle \underset{\text{by (4)}}{=} (\langle \vec{v}, a\vec{u} \rangle)^* \underset{\text{by (2)}}{=} (a \langle \vec{v}, \vec{u} \rangle)^* = a^* \langle \vec{v}, \vec{u} \rangle^*$$

$$\underset{\text{by (4)}}{=} a^* \langle \vec{u}, \vec{v} \rangle$$

So $\langle \vec{u}, a\vec{v} \rangle = a \langle \vec{u}, \vec{v} \rangle$

but $\langle a\vec{u}, \vec{v} \rangle = a^* \langle \vec{u}, \vec{v} \rangle$ for complex vector space

Why is this necessary to have the extra complex conjugation? For a complex vector space, both $\vec{u}$ and $\vec{v} = i\vec{u}$ are vectors in $\mathbb{V}$. By positivity, we must have both $\langle \vec{u}, \vec{u} \rangle > 0$ and $\langle \vec{v}, \vec{v} \rangle > 0$ (assuming $\vec{u} \neq \vec{0}$).

~~Check:~~

If $\langle \vec{u}, a\vec{v} \rangle = a \langle \vec{u}, \vec{v} \rangle$ and $\langle a\vec{u}, \vec{v} \rangle = a \langle \vec{u}, \vec{v} \rangle$, then $\langle \vec{v}, \vec{v} \rangle = \langle i\vec{u}, i\vec{u} \rangle = i^2 \langle \vec{u}, \vec{u} \rangle = -\langle \vec{u}, \vec{u} \rangle < 0$ if $\langle \vec{u}, \vec{u} \rangle > 0$. Violates positivity.

If $\langle \vec{u}, a\vec{v} \rangle = a \langle \vec{u}, \vec{v} \rangle$ and $\langle a\vec{u}, \vec{v} \rangle = a^* \langle \vec{u}, \vec{v} \rangle$, then $\langle \vec{v}, \vec{v} \rangle = \langle i\vec{u}, i\vec{u} \rangle = i\, i^* \langle \vec{u}, \vec{u} \rangle = i(-i) \langle \vec{u}, \vec{u} \rangle$
$= \langle \vec{u}, \vec{u} \rangle > 0$ as expected.

The most useful basis vectors $\vec{e}_i$ satisfy the following:

(1) Linearly independent: if basis is not linearly independent, can express one basis vector (say $\vec{e}_N$) as a linear combination of the others $\vec{e}_1, \ldots, \vec{e}_{N-1}$:

$$\vec{e}_N = \sum_{i=1}^{N-1} c_i \vec{e}_i$$

Equivalent to finding some values of c_i to such that

$$\sum_{i=1}^{N} c_i \vec{e}_i = 0$$

If only solution is $c_i = 0$ for all $i = 1, \dots, N$, then basis is linearly independent.

(2) Must <u>span</u> the vector space. Any vector $\vec{v}$ can be expressed as $\vec{v} = \sum_{i=1}^{N} v_i \vec{e}_i$

(3) Basis vectors are orthogonal $\langle \vec{e}_i, \vec{e}_j \rangle = 0$ for $i \neq j$

(4) Basis vectors are normalized to unity $\langle \vec{e}_i, \vec{e}_i \rangle = 1$.

Basis satisfying (3) & (4) is called <u>orthormal</u>.

$$\langle \hat{e}_i, \hat{e}_j \rangle = \delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases} \quad \text{(Kronecker Delta)}$$

Hats denote unit vectors (norm = 1)

The number of ~~unit~~ basis vectors, N, satisfying both (1) and (2) is the dimension of the vector space.

example: 3-dimensional space $\mathbb{R}^3$ has 3 basis vectors $\{\hat{e}_x, \hat{e}_y, \hat{e}_z\}$

The basis $\{\hat{e}_x, \hat{e}_y, \hat{e}_z, \hat{e}_4 = \hat{e}_x + \hat{e}_y\}$ doesn't satisfy (1) since $\hat{e}_4 = \hat{e}_x + \hat{e}_y$

The basis $\{\hat{e}_x, \hat{e}_y\}$ doesn't satisfy (2) since a vector in the z-direction cannot be expressed in terms of $\hat{e}_x, \hat{e}_y$ only.

Inner product in terms of components:

$$\langle \vec{u}, \vec{v} \rangle = \langle \Sigma_{i=1}^{N} u_i \hat{e}_i , \Sigma_{j=1}^{N} v_j \hat{e}_j \rangle$$

$\downarrow$ real vector space

$$= \Sigma_{i,j} u_i v_j \langle \hat{e}_i , \hat{e}_j \rangle = \Sigma_{i,j} u_i v_j \delta_{ij}$$

$$= \Sigma_{i=1}^{N} u_i v_i \quad \text{(usual dot product)}$$

$$\langle \vec{u}, \vec{v} \rangle = u_i^* v_i \quad \text{for complex vector space}$$

Matrices: In physics, matrices are often used as operators acting on a vector $\vec{v}$ to generate a new vector $\vec{v}'$:

$$\vec{v}' = M \vec{v} \quad \text{or in components} \quad v_i' = \Sigma_{j=1}^{N} M_{ij} v_j$$

(only consider $N \times N$ square matrices)

multiplying matrices together: $(AB)_{ij} = \Sigma_k A_{ik} B_{kj}$

Rotation matrices: An operator that rotates a vector can be represented by a matrix R that leaves the inner product invariant:

Want $\langle \vec{v}', \vec{w}' \rangle = \langle \vec{v}, \vec{w} \rangle$

where $\vec{v}' = R\vec{v}$ and $\vec{w}' = R\vec{w}$

Real vector space:

$$\langle \vec{v}', \vec{w}' \rangle = \sum_{i=1}^{N} v_i' w_i' = \Sigma_{ijk} R_{ij} v_j R_{ik} w_k$$

$$= \Sigma_{ijk} v_j (R^T)_{ji} R_{ik} w_k \overset{\text{want}}{=} \Sigma_i v_i w_i$$

$$= \langle \vec{v}, \vec{w} \rangle$$

This holds provided $\Sigma_i (R^T)_{ji} R_{ik} = \delta_{jk}$

or in Matrix notation $R^T R = \mathbb{1}$

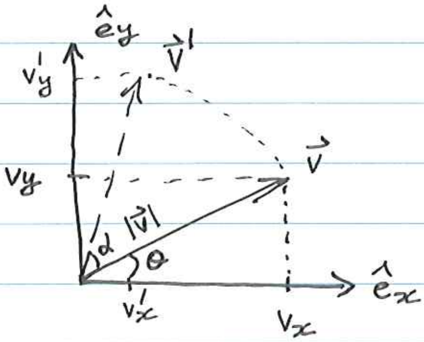
Note: the inverse of a matrix M is defined by $M^{-1}M = MM^{-1} = \mathbb{1}$

Therefore: rotation matrices on a real vector space, which leave the inner product invariant, are described by <u>orthogonal</u> matrices R, which satisfy $R^T R = \mathbb{1}$ or $R^T = R^{-1}$.

(For complex vector space: <u>unitary</u> matrix R (s.t. $R^\dagger = (R^*)^T = R^{-1}$) leaves inner prod. inv.

<u>Example</u>: rotations on $\mathbb{R}^2$ (real plane)

Consider a vector $\vec{v} = \begin{pmatrix} v_x \\ v_y \end{pmatrix}$ on the x-y plane



$$\vec{v} = \begin{pmatrix} v_x \\ v_y \end{pmatrix} = |\vec{v}| \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$$

Now rotate by an angle α to obtain a new vector $\vec{v}'$,

where $\vec{v}' = \begin{pmatrix} v_x' \\ v_y' \end{pmatrix} = |\vec{v}'| \begin{pmatrix} \cos(\theta+\alpha) \\ \sin(\theta+\alpha) \end{pmatrix}$

Rotation leaves the norm invariant, so $|\vec{v}'| = |\vec{v}|$.

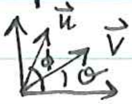
So $\vec{v}' = |\vec{v}| \begin{pmatrix} \cos\theta\cos\alpha - \sin\theta\sin\alpha \\ \sin\theta\cos\alpha + \cos\theta\sin\alpha \end{pmatrix}$

$$= |\vec{v}| \begin{pmatrix} \cos\alpha & -\sin\alpha \\ +\sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} = \underbrace{\begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}}_{\text{rotation matrix } R(\alpha)} \vec{v}$$

Check: $R(\alpha)^T R(\alpha) = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbb{1}$

as expected.

$R(\alpha)$ also leaves the inner product invariant; consider two vectors $\vec{u}, \vec{v}$.



$$\langle \vec{u}, \vec{v} \rangle = u_x v_x + u_y v_y = |\vec{u}||\vec{v}| (\cos\theta \cos\phi + \sin\theta \sin\phi)$$
$$= |\vec{u}||\vec{v}| \cos(\theta - \phi) \quad \text{(usual dot product formula)}$$

Now define
$$\vec{u}' = R(\alpha)\vec{u} = |\vec{u}'| \begin{pmatrix} \cos(\phi+\alpha) \\ \sin(\phi+\alpha) \end{pmatrix}$$
$$\vec{v}' = R(\alpha)\vec{v} = |\vec{v}'| \begin{pmatrix} \cos(\theta+\alpha) \\ \sin(\theta+\alpha) \end{pmatrix}$$

$$\langle \vec{u}', \vec{v}' \rangle = |\vec{u}'||\vec{v}'| \cos(\theta + \alpha - \phi - \alpha) = |\vec{u}||\vec{v}| \cos(\theta - \phi)$$
$$= \langle \vec{u}, \vec{v} \rangle \quad \text{as expected.}$$

Exponentiation of matrices: it can be useful to consider the quantity e^M where M is a matrix. What does it mean? When in doubt, Taylor expand!

Recall: Taylor expansion of a function $f(x)$ about $x=0$
$$f(x) = f(0) + f'(0)\,x + \frac{1}{2!} f''(0)\,x^2 + \frac{1}{3!} f'''(0)\,x^3 + \dots$$
For $f(x) = e^x$, we have
$$e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \dots = \sum_{n=0}^{\infty} \frac{1}{n!} x^n$$

Similarly, e^M is defined as

$$e^M = \mathbb{1} + M + \frac{1}{2!}M^2 + \frac{1}{3!}M^3 + \dots = \sum_{n=0}^{\infty} \frac{1}{n!} M^n$$

where $M^n = \underbrace{M M \cdots M}_{n \text{ times}}$ matrix multiplication.

The 2d rotation matrix $R(\alpha)$ can be expressed as

$$R(\alpha) = \exp(\alpha T) \quad \text{where } T = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \text{ is an antisymmetric } 2\times 2 \text{ matrix}$$

First, note that $T^2 = TT = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = -\mathbb{1}$

Then $T^{2n} = (-1)^n \mathbb{1}$ and $T^{2n+1} = (-1)^n T$

$$e^{\alpha T} = \sum_{n=0}^{\infty} \frac{1}{n!} \alpha^n T^n$$

$$= \underbrace{\left(\mathbb{1} + \frac{1}{2!}\alpha^2 T^2 + \frac{1}{4!}\alpha^4 T^4 + \dots\right)}_{\text{even } n \text{ terms}} + \underbrace{\left(\alpha T + \frac{1}{3!}\alpha^3 T^3 + \dots\right)}_{\text{odd } n \text{ terms}}$$

$$= \left(\mathbb{1} - \frac{1}{2!}\alpha^2 \mathbb{1} + \frac{1}{4!}\alpha^4 \mathbb{1} + \dots\right) + \left(\alpha T - \frac{1}{3!}\alpha^3 T + \frac{1}{5!}\alpha^5 T + \dots\right)$$

$$= \left(1 - \frac{1}{2!}\alpha^2 + \frac{1}{4!}\alpha^4 + \dots\right)\mathbb{1} + \left(\alpha - \frac{1}{3!}\alpha^3 + \frac{1}{5!}\alpha^5 + \dots\right) T$$

note: $\cos\alpha = 1 - \frac{1}{2!}\alpha^2 + \frac{1}{4!}\alpha^4 + \dots$ $\quad \sin\alpha = \alpha - \frac{1}{3!}\alpha^3 + \frac{1}{5!}\alpha^5 + \dots$

Taylor expansion of sin & cos

$$= \cos\alpha\, \mathbb{1} + \sin\alpha\, T$$

$$= \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} = R(\alpha)$$

Orthogonal matrices R (i.e. satisfying $R^T = R^{-1}$ or $R^T R = \mathbb{1}$) can be expressed as the exponential of an antisymmetric matrix ($T = -T^T$, or $T_{ij} = -T_{ji}$)

Rotations as complex numbers (in 2d only):

Review: a complex number has the form $z = x + iy$, where x is the real part, y is the imaginary part, and $i = \sqrt{-1}$.

Note: $i^2 = -1$, $i^3 = (i^2)i = -i$, $i^4 = (i)^2(i^2) = +1$.

Complex conjugation: $z^* = x - iy$

Magnitude: $|z| = \sqrt{z^* z} = \sqrt{x^2 + y^2}$

Inverse: $\frac{1}{z} = \frac{z^*}{|z|^2}$ $\left(\text{so } z \cdot \frac{1}{z} = z \frac{z^*}{|z|^2} = \frac{|z|^2}{|z|^2} = 1\right)$

Eulers formula: $e^{i\phi} = \cos\phi + i\sin\phi$

proof: Taylor expand both sides about $\phi = 0$.

$$e^{i\phi} = \sum_{n=0}^{\infty} \frac{1}{n!} i^n \phi^n = 1 + i\phi + \frac{1}{2}(i\phi)^2 + \frac{1}{3!}(i\phi)^3 + \frac{1}{4!}(i\phi)^4 + \frac{1}{5!}(i\phi)^5 + \dots$$

$$= \left(1 - \frac{1}{2!}\phi^2 + \frac{1}{4!}\phi^4 - \dots\right) + \left(i\phi - \frac{1}{3!} i\phi^3 + \frac{1}{5!} i\phi^5 - \dots\right)$$

$$\cos\phi = 1 - \frac{1}{2!}\phi^2 + \frac{1}{4!}\phi^4 - \dots$$

$$\sin\phi = \phi - \frac{1}{3!}\phi^3 + \frac{1}{5!}\phi^5 - \dots$$

so $e^{i\phi} = \cos\phi + i\sin\phi$. Setting $\phi = \pi$: $e^{i\pi} = -1$.

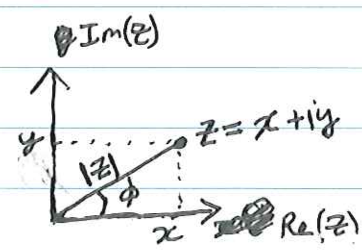
Then we can write: $z = |z| e^{i\phi}$ where ϕ is the phase or argument of z. (Also call $e^{i\phi}$ a "phase".)

$$z = |z| e^{i\phi} = |z|\cos\phi + i|z|\sin\phi \Rightarrow \begin{aligned} x &= |z|\cos\phi \\ y &= |z|\sin\phi \end{aligned}$$

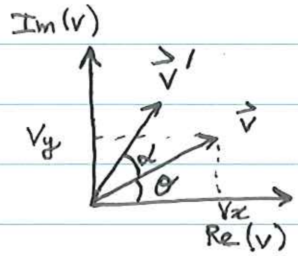
$$\Rightarrow \tan\phi = \frac{y}{x}$$

or $\phi = \arctan(y/x)$

Complex plane:



Rotations in 2d: instead of representing $\vec{v}$ as $\begin{pmatrix} v_x \\ v_y \end{pmatrix}$, we can represent it as a complex number $v = v_x + i v_y = |\vec{v}| e^{i\theta}$.



$$\vec{v}' = R(\alpha)\vec{v} = |\vec{v}'| \begin{pmatrix} \cos(\theta+\alpha) \\ \sin(\theta+\alpha) \end{pmatrix} \quad \text{and } |\vec{v}'| = |\vec{v}|$$

can be expressed as

$$v' = v_x' + i v_y' = |\vec{v}| e^{i(\theta+\alpha)}$$

$$v = |\vec{v}| e^{i\theta}$$

$$\Rightarrow \quad v' = e^{i\alpha} v$$

In this formulation, the rotation operator is given by a a phase: $R(\alpha) = e^{i\alpha}$. Magnitude of $\vec{v}$ unchanged since $|e^{i\alpha}| = 1$.

Same operator can have different representations:

2x2 orthogonal matrices equivalent to multiplication by a phase $e^{i\alpha}$.