Eigenvalues and eigenvectors

For a matrix M, $\vec{v}$ is an eigenvector of M if

$$M\vec{v} = \lambda \vec{v} \qquad (\text{and } \vec{v} \neq \vec{0})$$

where λ is a scalar. Then λ is ~~the~~ an eigenvalue of M.

Finding eigenvalues: want to solve $(M - \lambda \mathbb{1})\vec{v} = \vec{0}$.

Recall from linear algebra, the inverse A^{-1} of a Matrix A exists only if $\det(A) \neq 0$.

If $\det(M - \lambda\mathbb{1}) \neq 0$, then $\vec{v} = (M - \lambda \mathbb{1})^{-1}\vec{0} = \vec{0}$.
So we require

$$\det(M - \lambda \mathbb{1}) = 0 \qquad (\text{characteristic eqn})$$

Solve the characteristic equation to obtain λ. For an N-dimensional vector space, there will be at most N distinct ~~values of λ~~ eigenvalues.

Finding the eigenvectors: for each eigenvalue λ obtained, solve $M\vec{v} = \lambda\vec{v}$ to obtain eigenvectors $\vec{v}$. There will be N eigenvectors, some of which may have the same eigenvalue.

Example: Find eigen values & eigenvectors for $M = \begin{pmatrix} 1 & 2 \\ 2 & -2 \end{pmatrix}$

Eigen values: $0 = \det(M - \lambda \mathbb{1}) = \det \begin{pmatrix} 1-\lambda & 2 \\ 2 & -2-\lambda \end{pmatrix}$

$$= (1-\lambda)(-2-\lambda) - 4 = \lambda^2 + \lambda - 6 = (\lambda+3)(\lambda-2)$$

So $\lambda = 2$ and $\lambda = -3$ are the eigenvalues.

Eigenvectors: Solve $(M - \lambda \mathbb{1})\vec{v} = 0$ for each λ.

case of $\lambda = 2$:

$$(M - \lambda \mathbb{1})\vec{v} = \begin{pmatrix} -1 & 2 \\ 2 & -4 \end{pmatrix}\begin{pmatrix} V_x \\ V_y \end{pmatrix} = \begin{pmatrix} -V_x + 2V_y \\ 2V_x - 4V_y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\left.\begin{array}{r} -V_x + 2V_y = 0 \\ 2V_x - 4V_y = 0 \end{array}\right\} \Rightarrow V_x = 2V_y \Rightarrow \vec{v} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

case of $\lambda = -3$:

$$(M - \lambda \mathbb{1})\vec{v} = \begin{pmatrix} 4 & 2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} V_x \\ V_y \end{pmatrix} = \begin{pmatrix} 4V_x + 2V_y \\ 2V_x + V_y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\left.\begin{array}{r} 4V_x + 2V_y = 0 \\ 2V_x + V_y = 0 \end{array}\right\} \Rightarrow 2V_x = -V_y \Rightarrow \vec{v} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

The normalized eigenvectors are

$$\vec{v}_1 = \frac{1}{\sqrt{5}}\begin{pmatrix} 2 \\ 1 \end{pmatrix} \text{ for } \lambda = 2 \quad \text{and } \vec{v}_2 = \frac{1}{\sqrt{5}}\begin{pmatrix} 1 \\ -2 \end{pmatrix} \text{ for } \lambda = -3$$

Note: $\langle \vec{v}_1, \vec{v}_2 \rangle = 0$

So eigenvectors are orthogonal.

Useful facts about real symmetric matrices:

Note: A real matrix has entries that are purely real, so $M_{ij} = M_{ij}^* =$ a real number (or $M = M^*$)

A symmetric matrix is equal to its transpose i.e $M = M^T$ or $M_{ij} = M_{ji}$ in components.

Fact #1: the eigenvalues of a real symmetric matrix are real.

proof: Eigenvalue equation is $M\vec{v} = \lambda \vec{v}$

or in components $\sum_{j=1}^{N} M_{ij} v_j = \lambda v_j$ (eq. 1)

Take the complex conjugate of this equation:

$$\left(\sum_{j=1}^{N} M_{ij} v_j\right)^* = (\lambda v_j)^* \Rightarrow \sum_{j=1}^{N} M_{ij}^* v_j^* = \lambda^* v_i^*$$

If M is real & symmetric, then $M_{ij}^* \underset{\text{real.}}{\overset{\uparrow}{=}} M_{ij} \underset{\text{sym.}}{\overset{\uparrow}{=}} M_{ji}$

So then $$\sum_{j=1}^{N} M_{ji} v_j^* = \lambda^* v_i^* \qquad \text{(eq. 2)}$$

Now let's evaluate $\langle \vec{v}, M\vec{v} \rangle$. (Note: let $\vec{v}$ be a vector in a real or complex vector space.)

$$\langle \vec{v}, M\vec{v} \rangle = \sum_{i,j=1}^{N} v_i^* M_{ij} v_j \underset{\text{use eq. 1}}{=} \sum_{i=1}^{N} v_i^* \lambda v_i = \lambda \langle \vec{v}, \vec{v} \rangle$$

or we have

$$\langle \vec{v}, M\vec{v} \rangle = \sum_{i,j=1}^{N} v_i^* M_{ij} v_j \underset{\text{M real \& sym.}}{=} \sum_{i,j=1}^{N} v_i^* M_{ji}^* v_j =$$

$$= \sum_{j=1}^{N} v_j^* \lambda^* v_j = \lambda^* \langle \vec{v}, \vec{v} \rangle$$

use eq. 2

So we have shown that $\lambda\langle\vec{v},\vec{v}\rangle = \lambda^*\langle\vec{v},\vec{v}\rangle$ or
$(\lambda-\lambda^*)\langle\vec{v},\vec{v}\rangle = 0$.

Since $\langle\vec{v},\vec{v}\rangle > 0$ by positive-definiteness (and assuming $\vec{v}\neq\vec{0}$), we must have $\lambda = \lambda^* \Rightarrow \lambda$ is real.

Fact #2: For two distinct eigenvalues of M, the corresponding eigenvectors are orthogonal.

proof: Consider two eigenvectors $\vec{u}$ and $\vec{v}$, with eigenvalues λ_1, λ_2.
So $M\vec{u} = \lambda_1\vec{u}$ and $M\vec{v} = \lambda_2\vec{v}$.

Now $\langle\vec{u}, M\vec{v}\rangle = \langle\vec{u}, \lambda_2\vec{v}\rangle = \lambda_2\langle\vec{u},\vec{v}\rangle$.

But alternately we also have:

$$\langle\vec{u}, M\vec{v}\rangle = \sum_{i,j=1}^{N} u_i^* M_{ij} v_j = \sum_{i,j=1}^{N} M_{ji}^* u_i \, v_j = \sum_{j=1}^{N} (M\vec{u})_j^* v_j$$

$$= \langle M\vec{u}, \vec{v}\rangle = \lambda_1\langle\vec{u},\vec{v}\rangle$$

So $\lambda_1\langle\vec{u},\vec{v}\rangle = \lambda_2\langle\vec{u},\vec{v}\rangle$. The only way for this to be true (if $\lambda_1\neq\lambda_2$) is if $\langle\vec{u},\vec{v}\rangle = 0$.

So $\vec{u}$ and $\vec{v}$ are orthogonal.

Fact #3: An orthonormal basis of eigenvectors $\hat{v}_1, \dots, \hat{v}_N$ can be used to diagonalize M to the form $\begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_N \end{pmatrix}$, where the diagonal entries are the eigenvalues λ_a corresponding to each eigenvector $\hat{v}_a$ $(a = 1, 2, \dots, N)$.

Let's evaluate: $\langle \hat{v}_b, M\hat{v}_a \rangle = \lambda_a \langle \hat{v}_b, \hat{v}_a \rangle = \lambda_a \delta_{ab}$.

In component notation, this is

$$\langle \hat{v}_b, M\hat{v}_a \rangle = \sum_{i,j=1}^{N} (\hat{v}_b)_i^* M_{ij} (\hat{v}_a)_j = \lambda_a \delta_{ab}$$

Let's define a matrix $R = \left(\begin{pmatrix} \\ \hat{v}_1 \\ \\ \end{pmatrix} \begin{pmatrix} \\ \hat{v}_2 \\ \\ \end{pmatrix} \cdots \begin{pmatrix} \\ \hat{v}_N \\ \\ \end{pmatrix} \right)$ i.e. the columns of R are each of the orthonormal eigenvectors $\hat{v}_a$. R is an $N \times N$ matrix. The components of R are $R_{ia} = (\hat{v}_a)_i$.

Then $\langle \hat{v}_b, M\hat{v}_a \rangle = \sum_{i,j=1}^{N} R_{ib}^* M_{ij} R_{ja} = \lambda_a \delta_{ab}$

$$\Rightarrow \sum_{i,j=1}^{N} (R^\dagger)_{bi} M_{ij} R_{ja} = \lambda_a \delta_{ab}$$

where $R^\dagger$ is the Hermitian conjugate $(R^\dagger = (R^T)^*)$.

In matrix notation: $R^\dagger M R = \begin{pmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_N \end{pmatrix}$.

Note: $\langle \hat{v}_a, \hat{v}_b \rangle = \delta_{ab}$ since eigenvectors are orthonormal.

$$\langle \hat{v}_a, \hat{v}_b \rangle = \sum_{i=1}^{N} (\hat{v}_a)_i^* (\hat{v}_b)_i = \sum_{i=1}^{N} R_{ia}^* R_{ib} = \sum_{i=1}^{N} (R^\dagger)_{ai} R_{ib} = \delta_{ab}$$

or in matrix notation $R^\dagger R = \mathbb{1}$.

So the matrix R is <u>unitary</u>. $(R^\dagger = R^{-1})$

For the special case of a real vector space, the matrix R is also real (since the eigenvectors $\hat{v}_a$ have real components $(\hat{v}_a)_i$). So R is an orthogonal matrix.

example: Consider the matrix $M = \begin{pmatrix} 1 & 2 \\ 2 & -2 \end{pmatrix}$ (our previous example)

we had $\lambda_1 = 2$ $\qquad$ $\lambda_2 = -3$

$\hat{v}_1 = \frac{1}{\sqrt{5}} \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ $\qquad$ $\hat{v}_2 = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 \\ -2 \end{pmatrix}$ (real vector space)

then $R = \begin{pmatrix} \hat{v}_1 & \hat{v}_2 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix}$

check: $R^T M R = \left(\frac{1}{\sqrt{5}}\right)^2 \begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 2 & -2 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & -3 \end{pmatrix}$

eigenvalues

also $R^T R = \left(\frac{1}{\sqrt{5}}\right)^2 \begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

So R is orthogonal

Summary: real & complex vector spaces

	real	complex
For a vector $\vec{v}$, the components v_i are:	real	complex
Inner product $\langle \vec{u}, \vec{v} \rangle =$	$\langle \vec{v}, \vec{u} \rangle = \sum_{i=1}^{N} u_i v_i$	$\langle \vec{v}, \vec{u} \rangle^* = \sum_{i=1}^{N} u_i^* v_i$
Inner product is invariant under ______ rotations:	orthogonal $(R^T R = \mathbb{1})$	unitary $(R^\dagger R = \mathbb{1})$

Note: "$\dagger$" is the Hermitian conjugate, which is the combined complex conjugate and transpose of a matrix.

$$M^\dagger = (M^*)^T = (M^T)^*$$

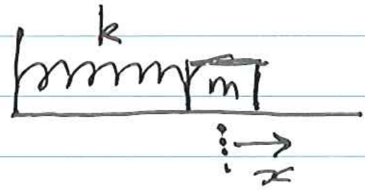
A Hermitian matrix is equal to its Hermitian conjugate (so $M = M^\dagger$ or in components $M_{ij} = M_{ji}^*$)

Real symmetric matrices (associated with physical systems described by real vector spaces) have (1) real eigenvalues, and (2) are diagonalized by orthogonal rotations.

Hermitian matrices (associated with physical systems described by complex vector spaces) have (1) real eigenvalues, and (2) are diagonalized by unitary rotations.

Application: coupled harmonic oscillators

First, consider the simple harmonic oscillator, e.g. a mass m attached to a spring, with spring constant k.



x is the (one-dim.) displacement from equilibrium.

The potential energy of the spring is $V = \frac{1}{2}kx^2$.
Newton's 2nd law: $F = ma$. The force is

$$F = -\frac{dV}{dx} = -kx$$

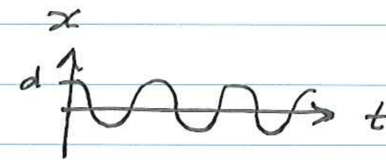
So the equation of motion is $\ddot{x} = -\frac{k}{m}x$ or $\ddot{x} = -\omega^2 x$ where $\omega = \sqrt{k/m}$ is the frequency of oscillation and "$\cdot$" $= d/dt$ (time-derivative).

The solution is $x(t) = A\sin(\omega t) + B\cos(\omega t)$, where A, B are undetermined constants. You determine A, B by using the initial conditions. e.g. suppose at time $t=0$ the system is at rest with displacement d. Then

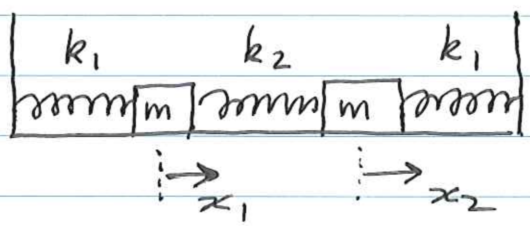
$$x(t=0) = d \quad \text{and} \quad \dot{x}(t=0) = 0$$

$$\Rightarrow \quad B = d \quad \text{and} \quad A = 0$$

So the solution is $x(t) = d\cos(\omega t)$



<u>Coupled harmonic oscillator</u> : Consider a system with two masses (of equal mass m) with spring attached as follows:



x_1 and x_2 are the displacements from equilibrium.

Potential energy: $V = \underbrace{\frac{1}{2}k_1 x_1^2}_{\text{Spring 1}} + \underbrace{\frac{1}{2}k_2(x_1 - x_2)^2}_{\text{Spring 2}} + \underbrace{\frac{1}{2}k_1 x_2^2}_{\text{Spring 3}}$

Newton's 2nd law:

$$m\ddot{x}_1 = F_1 = -\frac{\partial V}{\partial x_1} = -\left(k_1 x_1 + k_2(x_1 - x_2)\right)$$

$$m\ddot{x}_2 = F_2 = -\frac{\partial V}{\partial x_2} = -\left(-k_2(x_1 - x_2) + k_1 x_2\right)$$

We can write the 2nd law as a matrix equation:

$$\begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} = -\begin{pmatrix} \frac{k_1+k_2}{m} x_1 - \frac{k_2}{m} x_2 \\ -\frac{k_2}{m} x_2 \quad \frac{k_1+k_2}{m} x_2 \end{pmatrix} = -\underbrace{\begin{pmatrix} \frac{k_1+k_2}{m} & -k_2/m \\ -k_2/m & \frac{k_1+k_2}{m} \end{pmatrix}}_{\text{define this as } \mathcal{U}.} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Define a two-component vector $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$.

Then the 2nd law is $\ddot{\vec{x}} = -\mathcal{U}\vec{x}$.

The matrix $\mathcal{U}$ has the form $\mathcal{U} = \begin{pmatrix} a & -b \\ -b & a \end{pmatrix}$, where $a = \frac{k_1+k_2}{m}$, $b = \frac{k_2}{m}$.
Note: $\mathcal{U}$ is a real symmetric matrix.

Find the eigenvalues and eigenvectors of $\mathcal{U}$:

Eigenvalues: $0 = \det(\mathcal{U} - \lambda \mathbb{1}) = \det\begin{pmatrix} a-\lambda & -b \\ -b & a-\lambda \end{pmatrix} = (a-\lambda)^2 - b^2$

$$\Rightarrow \quad \lambda^2 - 2a\lambda + a^2 - b^2 = 0$$

$$\lambda = \frac{1}{2}\left[2a \pm \sqrt{4a^2 - 4(a^2-b^2)}\right] = a \pm b$$

So the eigenvalues are $\lambda = \begin{cases} k_1/m \\ \frac{k_1+2k_2}{m} \end{cases}$

~~Eigenvectors: let's call the eigenvectors $\hat{v}$ and $\hat{w}$~~

~~(i) case: $\lambda = \frac{k_1}{m} = \omega_1^2$,~~

Let's call the eigenvalues $\omega_1^2 = k_1/m$ and $\omega_2^2 = \frac{k_1+2k_2}{m}$
since they will be related to the frequencies.

Eigenvectors: Let's call the eigenvectors $\vec{v}$ and $\vec{w}$.

(i) case $\lambda = \omega_1^2 = k_1/m$:

$$(\mathcal{U} - \lambda \mathbb{1})\vec{v} = \begin{pmatrix} a-\omega_1^2 & -b \\ -b & a-\omega_1^2 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} k_2/m & -k_2/m \\ -k_2/m & +k_2/m \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \frac{k_2}{m} v_1 - \frac{k_2}{m} v_2 = 0 \quad \Rightarrow \quad v_1 = v_2 \quad \Rightarrow \quad \vec{v} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

(ii) case $\lambda = \omega_2^2 = \frac{k_1 + k_2}{m}$:

$$(\mathcal{U} - \lambda \mathbb{1})\vec{w} = \begin{pmatrix} a-\omega_2^2 & -b \\ -b & a-\omega_2^2 \end{pmatrix}\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} -k_2/m & -k_2/m \\ -k_2/m & -k_2/m \end{pmatrix}\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \frac{k_2}{m} w_1 = -\frac{k_2}{m} w_2 \quad \Rightarrow \quad w_1 = -w_2 \Rightarrow \vec{w} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Normalize the eigenvectors to unity:

$$\hat{v} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \hat{w} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Physical interpretation:

$\hat{v}$ is when $x_1 = x_2$; masses move in the same direction

$\hat{w}$ is when $x_1 = -x_2$; masses move in opposite direction.

Rotation matrix: $R = (\hat{v}\ \hat{w}) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$

check:
$$R^T \mathcal{U} R = \left(\frac{1}{\sqrt{2}}\right)^2 \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} a & -b \\ -b & a \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} a-b & 0 \\ 0 & a+b \end{pmatrix}$$
$$= \begin{pmatrix} \omega_1^2 & 0 \\ 0 & \omega_2^2 \end{pmatrix}$$

$$R^T R = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbb{1}$$

The original system of coupled equations can be simplified into a system of uncoupled equations.

Newton's 2nd law $\ddot{\vec{x}} = -\mathcal{U}\vec{x}$.

Define a new vector $\vec{x}' = R^T\vec{x}$ (or equivalently $\vec{x} = R\vec{x}'$).

Then $\ddot{\vec{x}}' = R^T\ddot{\vec{x}} = R^T(-\mathcal{U}\vec{x}) = -R^T\mathcal{U}R\vec{x}'$

$$\text{So} \quad \ddot{\vec{x}}' = -\begin{pmatrix} \omega_1^2 & 0 \\ 0 & \omega_2^2 \end{pmatrix}\vec{x}'$$

In terms of $\vec{x}'$, the system appears as two uncoupled simple harmonic oscillators with equations of motion

$$\ddot{x}_1' = -\omega_1^2 x_1 \quad \text{and} \quad \ddot{x}_2' = -\omega_2^2 x_2$$

The solutions are:
$$x_1'(t) = A_1 \sin(\omega_1 t) + B_1 \cos(\omega_1 t)$$
$$x_2'(t) = A_2 \sin(\omega_2 t) + B_2 \cos(\omega_2 t)$$

The variables x_1', x_2' are called the normal modes, and ω_1 and ω_2 are the normal frequencies.

The solutions for $x_1(t)$ and $x_2(t)$ are given by

$$\vec{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = R\vec{x}' = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} A_1\sin(\omega_1 t) + B_1\cos(\omega_1 t) \\ A_2\sin(\omega_2 t) + B_2\cos(\omega_2 t) \end{pmatrix}$$

Coefficients A_1, B_1, A_2, B_2 are determined by initial conditions.
e.g. suppose $x_1(t=0) = d$ and $x_2(t=0) = \dot{x}_1(t=0) = \dot{x}_2(t=0) = 0$.

$$\text{then } \vec{x}(0) = \begin{pmatrix} d \\ 0 \end{pmatrix} \text{ and } \dot{\vec{x}}(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Then $A_1 = A_2 = 0$ and $B_1 = B_2 = d/\sqrt{2}$.

<u>Example</u>: find eigenvalues & eigenvectors of $M = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix}$

Eigenvalues: $0 = \det(M - \lambda \mathbb{1}) = \det \begin{pmatrix} 1-\lambda & 0 & 1 \\ 0 & 2-\lambda & 0 \\ 1 & 0 & 1-\lambda \end{pmatrix}$

$$= (1-\lambda)^2(2-\lambda) - (2-\lambda) = ((1-\lambda)^2 - 1)(2-\lambda)$$

$$= (2-\lambda)(\cancel{1} - 2\lambda + \lambda^2 \cancel{-1}) = (2-\lambda)(2-\lambda)(-\lambda)$$

$$= -(\lambda-2)^2 \lambda = 0$$

$$\Rightarrow \quad \lambda = 0 \quad \text{and} \quad \lambda = 2 \text{ (double root)}$$

If the characteristic equation has a repeated root, this is called a <u>degenerate</u> eigenvalue.

Eigenvectors:

case $\lambda = 0$: $(M - \lambda \mathbb{1})\vec{u} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} u_x \\ u_y \\ u_z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$$\Rightarrow \quad u_x + u_z = 0 \quad \Rightarrow \quad u_x = -u_z$$

and $u_y = 0$

$$\Rightarrow \quad \vec{u} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

case $\lambda = 2$: $(M - \lambda \mathbb{1})\vec{v} = \begin{pmatrix} -1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$\Rightarrow \quad v_x = v_z \quad$ and $\quad 0 \, v_y = 0 \Rightarrow$ any v_y is allowed.

We have more freedom to choose the eigenvectors for a degenerate eigenvalue. Since this is a double root, there are two linearly independent eigenvectors.

e.g. pick $\vec{v} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$. To find the remaining eigenvector $\vec{w}$, find the vector that satisfies $\langle \vec{u}, \vec{w} \rangle = \langle \vec{v}, \vec{w} \rangle = 0$

$$\Rightarrow \vec{w} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.$$

So the three orthonomal eigenvectors are

$$\hat{u} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\-1\end{pmatrix}, \quad \hat{v} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\1\end{pmatrix}, \quad \hat{w} = \begin{pmatrix}0\\1\\0\end{pmatrix}$$

but choice of $\hat{v}, \hat{w}$ is not unique since any linear combination $a\hat{v} + b\hat{w}$ is also an eigenvector ~~for the~~ with eigenvalue $\lambda = 2$.

Lastly, one can check that the rotation matrix diagonalizing M is given by

$$R = \begin{pmatrix}\hat{u} & \hat{v} & \hat{w}\end{pmatrix} = \begin{pmatrix}1/\sqrt{2} & 1/\sqrt{2} & 0\\ 0 & 0 & 1\\ -1/\sqrt{2} & 1/\sqrt{2} & 0\end{pmatrix}.$$

$$R^T M R = \begin{pmatrix}0 & 0 & 0\\ 0 & 2 & 0\\ 0 & 0 & 2\end{pmatrix}$$ as expected.