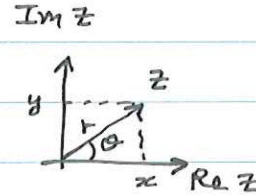
Complex analysis

Review of complex numbers



$$z = \underbrace{x + iy}_{\text{Cartesian form}} = \underbrace{r e^{i\theta}}_{\text{Polar form}} = r(\cos\theta + i\sin\theta)$$

real part $\quad x = \mathrm{Re}(z) = r\cos\theta$

imaginary part $\quad y = \mathrm{Im}(z) = r\sin\theta$

magnitude $\quad r = |z| = \sqrt{x^2+y^2}$

argument $\quad \theta = \arg(z) = \arctan(y/x)$

complex conjugate: $z^* = x - iy = r e^{-i\theta}$

Then $|z|^2 = z^* z = z z^* = r^2 = x^2 + y^2$

Usual rules for arithmetic apply to complex numbers

$z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$

addition: $z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2)$

subtraction: $z_1 - z_2 = (x_1 - x_2) + i(y_1 - y_2)$

multiplication: $z_1 z_2 = (x_1 + iy_1)(x_2 + iy_2) = (x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1)$

$= r_1 r_2 e^{i(\theta_1+\theta_2)}$

division: $z_1/z_2 = \dfrac{z_1}{z_2}\dfrac{z_2^*}{z_2^*} = \dfrac{(x_1+iy_1)(x_2-iy_2)}{x_2^2+y_2^2}$

$$= \left(\frac{x_1x_2 + y_1y_2}{x_2^2+y_2^2}\right) + i\left(\frac{x_2y_1 - x_1y_2}{x_2^2+y_2^2}\right)$$

$$= \frac{r_1}{r_2} e^{i(\theta_1-\theta_2)}$$

Euler's formula: $e^{i\theta} = \cos\theta + i\sin\theta$

Can express trig functions as $\quad \cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta})$

$\sin\theta = \frac{1}{2i}(e^{i\theta} - e^{-i\theta})$

Similar to hyperbolic trig functions: $\quad \cosh x = \frac{1}{2}(e^{x} + e^{-x})$

$\sinh x = \frac{1}{2}(e^{x} - e^{-x})$

Functions of complex variables

A complex function f takes a complex number $z=x+iy$ and returns another complex number $f(z)$.

Can write $f(z) = \underbrace{u(x,y)}_{Re(f(z))} + i\underbrace{v(x,y)}_{Im(f(z))}$

Example: $f(z) = e^{z} = e^{x+iy} = e^{x}(\cos y + i\sin y)$

So $u = e^{x}\cos y$, $v = e^{x}\sin y$

Example: $f(z) = \ln z = \ln(r e^{i\theta}) = \ln r + \ln(e^{i\theta})$

$= \ln r + i\theta = \ln|z| + i\arg(z)$

note: z is the same for $\theta \to \theta \pm 2\pi n$ $(n = 1,2,\ldots)$ since $e^{2\pi i n} = 1$; but under this change in θ, $\ln z = \ln r + i\theta \to \ln r + i\theta \pm 2\pi i n$ is not the same.

For $\ln z$, need to specify the allowed range for $\theta = \arg(z)$ (called the branch). The principle branch is any branch that includes $\theta = 0$.

e.g. take $z = -2i = 2e^{i\theta}$ where $\theta = -\frac{\pi}{2} \pm 2\pi n$.

can specify $0 \leq \theta < 2\pi \Rightarrow \theta = \frac{3\pi}{2} \Rightarrow \ln z = \ln 2 + \frac{3\pi}{2} i$

or can have $-\pi \leq \theta < \pi \Rightarrow \theta = -\frac{\pi}{2} \Rightarrow \ln z = \ln 2 - \frac{\pi}{2} i$

Need to be careful when evaluating $\ln z$ since depends on choice for allowed range of $\theta = \arg(z)$.

Complex roots

An equation of the form $z^m = c$, where $m = 1, 2, \ldots$ and c is a complex number, has m distinct solutions (roots) for z.

Express: $z = re^{i\theta}$ and $c = |c|e^{i\arg(c)}$

Then $z^m = r^m e^{im\theta} = |c|e^{i\arg(c)}$

The roots z have: $r = |c|^{1/m}$

$$m\theta = \arg(c) + 2\pi n \qquad (n = 0, \pm 1, \pm 2, \ldots)$$

$$\theta = \frac{1}{m}\arg(c) + \frac{2\pi n}{m}$$

restricting $0 \le \theta < 2\pi$, there will be exactly m values of n for which $0 \le \theta < 2\pi$. These values of n give the values of $\theta = \theta_1, \theta_2, \ldots, \theta_m$

So the roots are $z = |c|^{1/m}e^{i\theta_1}, |c|^{1/m}e^{i\theta_2}, \ldots |c|^{1/m}e^{i\theta_m}$

example: $z^3 = 1 \Rightarrow r^3 e^{3i\theta} = 1 \Rightarrow r = 1$

$3\theta = \arg(1) + 2\pi n \Rightarrow \theta = \frac{2\pi n}{3} = \underset{n=0}{0}, \underset{n=1}{\frac{2\pi}{3}}, \underset{n=2}{\frac{4\pi}{3}}$

$z^3 = 1$ has roots $\{1, e^{i2\pi/3}, e^{i4\pi/3}\}$

example: $z^2 = 1 + i \qquad r^2 e^{2i\theta} = 1 + i \cancel{\Rightarrow r = \sqrt{1+1} = \sqrt{2}}$

$r = |1+i|^{1/2} = \sqrt{1+1} = \sqrt{2}$

$2\theta = \arg(1+i) + 2\pi n$

$\theta = \frac{1}{2}\frac{\pi}{4} + \pi n = \underset{n=0}{\frac{\pi}{8}}, \underset{n=1}{\frac{9\pi}{8}}$

So $z = \sqrt{2}e^{i\pi/8}, \sqrt{2}e^{9i\pi/8}$