Complex analysis review problems

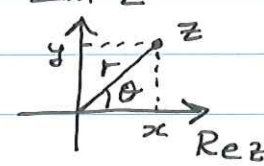
① Cartesian & polar form: $z = x + iy = r e^{i\theta}$

example: $-5i = 5e^{3\pi i/2}$

cartesian polar

example: $2e^{3i\pi/4} = 2\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right) = -\sqrt{2} + i\sqrt{2}$

polar cartesian



② argument: $\arg(z) = \theta = \arctan(y/x)$

magnitude: $r = |z| = \sqrt{x^2+y^2}$

example: $\arg(-i) = 3\pi/2$

③ ln of complex numbers:

$$\ln z = \ln(re^{i\theta}) = \ln r + \ln e^{i\theta} = \ln r + i\theta$$
$$= \ln|z| + i\arg(z)$$

example: $\ln(i) = \ln|i| + i\arg(i)$
$= \ln(1) + i\pi/2 = i\pi/2$

④ Complex roots:

example: $z^3 = 4\sqrt{2} + 4\sqrt{2}\,i$

express both sides in Polar form

$$z^3 = r^3 e^{3i\theta}$$
$$4\sqrt{2} + 4\sqrt{2}\,i = 8e^{i\pi/4}$$

Then $r^3 = 8$ or $r = 2$

and $3\theta = \pi/4 + 2\pi n \qquad (n = 0, 1, 2, ..)$

So $\theta = \frac{\pi}{12}, \frac{\pi}{12} + \frac{2\pi}{3}, \frac{\pi}{12} + 2\frac{2\pi}{3}$

$= \frac{\pi}{12}; \frac{3\pi}{4}, \frac{17\pi}{12}$ ← only 3 values of θ between 0 & 2π

$$z = 2e^{i\pi/12}, 2e^{3i\pi/4}, e^{17i\pi/12}$$

3 roots because original equation was z^3.

(5) Analytic functions: $f(z) = u(x,y) + iv(x,y)$

Cauchy-Riemann relations: $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

example: $f(z) = \sin z$. find u & v.

$$f(z) = \sin(x+iy) = \frac{1}{2i}\left(e^{i(x+iy)} - e^{-i(x+iy)}\right)$$

$$= \frac{1}{2i}\left(e^{ix}e^{-y} - e^{-ix}e^{y}\right)$$

$$= \frac{1}{2i}\left(e^{-y}(\cos x + i\sin x) - e^{y}(\cos x - i\sin x)\right)$$

$$= \frac{1}{2}\sin x\,(e^{y} + e^{-y}) + i\cos x\,(e^{y} - e^{-y})/2$$

So $u(x,y) = \sin x \, \frac{e^{y} + e^{-y}}{2} = \sin x \cosh y$

$v(x,y) = \cos x \, \frac{e^{y} - e^{-y}}{2} = \cosh x \sinh y$

Can plug into C-R relations to check analiticity.

⑥ Complex integrals $\int_{z_1}^{z_2} dz\, f(z)$ along a path P.

example: evaluate $f(z)$ from $z_1 = 0$ to $z_2 = 1+i$ along Path P defined by $y = x$.

$$\int_0^{1+i} dz\, z = \int_0^{1+i} (dx + i\,dy)(x+iy)$$

Now use $y = x \rightarrow dy = dx$

$$\int_0^{1+i} dz\, z = \int_{x=0}^{x=1} (1+i)\,dx\,(1+i)\,x = (1+i)^2 \frac{1}{2} = i$$

or can integrate directly: $\int_0^{1+i} dz\, z = \frac{(1+i)^2}{2} = i$

⑦ Poles & essential singularities

- $\frac{1-\cos z}{z}$ → removable singularity at $z=0$

$$\frac{1-\cos z}{z} = \frac{1-(1-\frac{1}{2!}z^2+\dots)}{z} = \frac{\frac{1}{2}z^2+\dots}{z} = \frac{1}{2}z+\dots$$

doesn't blow up for $z \to 0$.

- $\frac{z+2}{z^3+1}$ → poles

3 simple poles corresponding to the roots of $z^3 = -1$.

→ $z = e^{i\pi/3}, e^{5i\pi/3}, -1$ simple poles.

- $z^3 e^{1/z}$ essential singularity

⑧ Contour integrals ☆☆☆ Very important

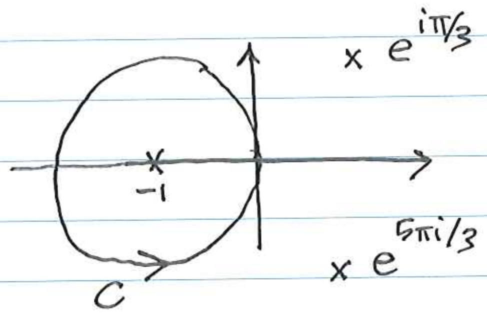
- Evaluate $\oint_C dz \, \frac{1-\cos z}{z}$ where C is circle $|z|=1$

Since function is analytic everywhere (the pole at $z=0$ is removable), then

$$\oint_C dz \, \frac{1-\cos z}{z} = 0 \quad \text{by Cauchy's theorem.}$$

- Evaluate $\oint_C dz \, \frac{z+2}{z^3+1}$ for C defined by $|z+1|=1$, a circle centered at $z=-1$ of radius 1.

Poles at $z = e^{i\pi/3}, -1, e^{5\pi i/3}$



only pole at $z=-1$ enclosed within C

Let $f(z) = \frac{z+2}{z^3+1} = \frac{g(z)}{h(z)}$

$$\oint_C dz \, \frac{z+2}{z^3+1} = 2\pi i \operatorname{Res} f(-1) = 2\pi i \frac{g(-1)}{h'(-1)} = 2\pi i \frac{1}{3}$$

$$= + \frac{2\pi i}{3}$$

- Evaluate $\oint_C dz \, \underbrace{z^3 e^{1/z}}_{f(z)}$ for C defined by $|z|=1$

essential singularity at $z=0$. Use Laurent expansion.

$$z^3 e^{1/z} = z^3\left(1 + \frac{1}{z} + \frac{1}{2!}\frac{1}{z^2} + \frac{1}{3!}\frac{1}{z^3} + \frac{1}{4!}\frac{1}{z^4} + \dots\right)$$

$$= z^3 + z^2 + \frac{1}{2}z + \frac{1}{6} + \frac{1}{24}\frac{1}{z} + \dots$$

coefficient c_{-1} is $\frac{1}{24}$.

$$\text{Res } f(0) = c_{-1} = \frac{1}{24}$$

So $\oint_C dz\, z^3 e^{1/z} = 2\pi i \text{ Res } f(0) = \underline{\frac{\pi i}{12}}$

(9) <u>Trig integrals</u> ☆☆☆ <u>Very important</u>

evaluate $\int_0^{2\pi} d\theta \frac{1}{5+3\sin\theta}$. $\quad z = e^{i\theta}$ (note $|z|=1$) $\quad dz = i e^{i\theta} d\theta = i z\, d\theta$

So $\int_0^{2\pi} d\theta = \oint_C \frac{dz}{iz}$ where C is circle $|z|=1$.

And $\sin\theta = \frac{1}{2i}(e^{i\theta} - e^{-i\theta}) = \frac{1}{2i}(z - \frac{1}{z})$

So $\int_0^{2\pi} d\theta \frac{1}{5+3\sin\theta} = \oint_C \frac{dz}{iz} \frac{1}{5+\frac{3}{2i}(z-\frac{1}{z})}$

$$= \oint \frac{dz}{i} \frac{2i}{10iz+3z^2-3} = \oint dz \frac{2}{3z^2+10iz-3}$$

Find roots of denominator: $3z^2+10iz-3=0$

$$z = \frac{1}{2\cdot 3}\left(-10i \pm \sqrt{-100+4\cdot 9}\right) = \frac{1}{6}(-10i \pm 8i)$$

$$= -3i, \ -\frac{i}{3}$$ only $z = -\frac{i}{3}$ is within C.

$$\oint dz \underbrace{\frac{2}{3z^2+10z-3}}_{f(z)} = 2\pi i \text{ Res } f(-\tfrac{i}{3})$$

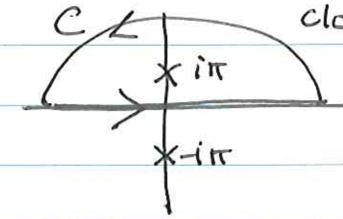
$$\text{Res}\, f\left(-\tfrac{i}{3}\right) = \lim_{z \to -\frac{i}{3}} (z + i/3)\, \frac{2}{3z^2 + 10iz - 3} = \frac{2}{8i}$$

$$= \frac{1}{4i}$$

So $$\int_0^{2\pi} d\theta\, \frac{1}{5 + 3\sin\theta} = \frac{1}{4i} \times 2\pi i = \underline{\frac{\pi}{2}}$$

10 Real Integrals ☆☆☆ Very important

$$\int_{-\infty}^{\infty} dx\, \frac{1}{x^2 + \pi^2} = \oint_C dz\, \frac{1}{z^2 + \pi^2}$$



close contour at ∞

$f(z) = \dfrac{1}{z^2 + \pi^2}$ has poles at $z = i\pi$ and $z = -i\pi$.

Only the pole at $z = i\pi$ is enclosed.

$$\text{Res}\, f(i\pi) = \lim_{z \to i\pi} (z - i\pi)\, \frac{1}{z^2 + \pi^2} = \frac{1}{2i\pi}$$

So $$\int_{-\infty}^{\infty} dx\, \frac{1}{x^2 + \pi^2} = \oint_C dz\, \frac{1}{z^2 + \pi^2} = 2\pi i \,\text{Res}\, f(i\pi) = \underline{1}$$

(Same answer closing the contour below.)