PHYS 3090: Homework 1 (due Friday Sept. 18)

Problem 1 (5 points): Let $\vec{u} = \begin{pmatrix} 2 \\ -i \end{pmatrix}$, $\vec{v} = \begin{pmatrix} 2-i \\ 2i \end{pmatrix}$, and $a = 2i - 2$. Compute $\langle \vec{u}, \vec{v} \rangle$ and $\langle a\vec{u}, \vec{v} \rangle$.

Problem 2 (15 points): Find the eigenvalues λ_1, λ_2 for the Hermitian matrix $M = \begin{pmatrix} 3 & 2i \\ -2i & 0 \end{pmatrix}$. Find any matrix R such that $R^\dagger M R = \mathrm{diag}(\lambda_1, \lambda_2)$. Show that R is unitary.

Problem 3 (20 points): Consider a vector $\vec{v} = \begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix}$ in $\mathbb{R}^3$ (3d space).

- Find the 3×3 rotation matrix $R_z(\alpha)$ that rotates $\vec{v}$ about the $\hat{e}_z$ axis by an angle α. Similarly, find the rotation matrices $R_x(\alpha)$ and $R_y(\alpha)$ that rotate $\vec{v}$ about the $\hat{e}_x$ and $\hat{e}_y$ axes, respectively. **(10 points)**
- Show that two rotations about the same axis commute by proving $R_z(\alpha)R_z(\theta) = R_z(\theta)R_z(\alpha)$. Show that two rotations about different axes do not commute. (Note: Two matrices A and B **commute** if $AB = BA$.) **(10 points)**

Problem 4 (5 points): The equation for a simple harmonic oscillator is $\ddot{x} = -\omega^2 x$, where ω is the frequency of the oscillator. The most general solution is

$$x(t) = a\cos(\omega t) + b\sin(\omega t)\,, \tag{1}$$

where a and b are real numbers.
Show that the solution

$$x(t) = \mathrm{Re}\big(c\, e^{i\omega t}\big)\,, \tag{2}$$

where c is a complex number, is equivalent to Eq. (1) by relating c to a and b.

Problem 5 (25 points): The equation of motion for a 3d simple harmonic oscillator is $\ddot{\vec{r}} = -\omega^2 \vec{r}$, where $\vec{r}(t) = \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix}$. Similar to problem 5, the most general solution can be expressed as

$$\vec{r}(t) = \mathrm{Re}\big(\vec{c}\, e^{i\omega t}\big)\,, \tag{3}$$

where $\vec{c}$ is an arbitrary three-component complex vector.

- Show that Eq. (3) solves the equation of motion for any vector $\vec{c}$. **(5 points)**

Next, we will see how different choices for $\vec{c}$ leads to different physical behaviors for $\vec{r}(t)$. In particular, we focus on the connection between rotations and angular momentum, which appears in many physical contexts.

- For the rotation matrix

$$R_x(\alpha) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{pmatrix}, \tag{4}$$

  determine the eigenvalues $(\lambda_1, \lambda_2, \lambda_3)$ and eigenvectors $(\vec{u}_1, \vec{u}_2, \vec{u}_3)$. **(5 points)**

- Evaluate $\vec{r}(t)$ separately for each eigenvector by setting $\vec{c} = \vec{u}_i$, as follows (**5 points**):

$$\vec{r}_1(t) = \mathrm{Re}\big(\vec{u}_1\, e^{i\omega t}\big)\,, \quad \vec{r}_2(t) = \mathrm{Re}\big(\vec{u}_2\, e^{i\omega t}\big)\,, \quad \vec{r}_3(t) = \mathrm{Re}\big(\vec{u}_3\, e^{i\omega t}\big)\,. \tag{5}$$

- Evaluate the angular momentum vector $\vec{L} = \vec{r} \times \vec{p}$ for each of the solutions obtained above (recall $\vec{p} = m\dot{\vec{r}}$, where m is the mass). Show that the solutions have either $\vec{L} = 0$ or $\vec{L}$ oriented along the x axis. **(5 points)**

- Evaluate $R_x(\alpha)\vec{r}_i(t)$ for each solution $i = 1, 2, 3$. **(5 points)**