PHYS 3090: Homework 2 (due Wed. Sept. 28)

Problem 1: Identify the singular points in the following functions and demonstrate whether they are poles, essential singularities, or removable singularities. For any poles or essential singularities, determine their residues.

(a) $f(z) = \cos(z + 1/z)$ **(3 points)**

(b) $f(z) = \frac{z^3+6z^2+5z-12}{3z^2-6z+3}$ **(3 points)**

(c) $f(z) = \frac{z^2-4}{z+2}$ **(3 points)**

(d) $f(z) = \cot(z)/z^2$ **(3 points)**

The following problem derives a useful formula for computing residues.

Problem 2: Consider the function $f(z) = g(z)/h(z)$, where $g(z)$ is entire.

(a) Show that if $f(z)$ has a simple pole at $z = a$, then $\mathrm{Res}\, f(a) = g(a)/h'(a)$. Hint: Taylor expand $h(z)$. **(3 points)**

(b) Now let's generalize this result to a pole of arbitrary order. Show that if $f(z)$ has a pole of order n at $z = a$, then we have the following formula **(3 points)**

$$\mathrm{Res}\, f(a) = \frac{n\, g^{(n-1)}(a)}{h^{(n)}(a)}\,. \tag{1}$$

(c) For the function $f(z) = \frac{e^z-1}{e^z+1}$, determine the locations and orders for all poles and compute their residues. **(3 points)**

Problem 3: Compute the following contour integrals $\oint_C dz\, f(z)$, where

(a) $f(z) = \frac{1}{z^2+1}$, where C is the circle $|z - i| = 1$ **(3 points)**

(b) $f(z) = \frac{1}{z^4+1}$, where C is the rectangle with corners at $z = \pm 2i$ and $z = 2 \pm 2i$ **(3 points)**

(c) $f(z) = \tan(z)$, where C is the circle $|z| = 5$ **(3 points)**

This problem is a bit challenging. Don't get confused: there is a simple pole at $z = 1$, but this is not the singular point you should be concerned with.

Problem 4: Compute $\oint_C dz\, \frac{e^{1/z}}{1-z}$ where C is the circle $|z| = 0.1$. Hint: you will need to use the formula for an infinite geometric series: $\sum_{n=0}^{\infty} z^n = \frac{1}{1-z}$ for $|z| < 1$. **(3 points)**

Picard's theorem is a remarkable result which says that if $f(z)$ has an essential singularity at $z = a$, then within **any** *finite neighborhood of a, no matter how small, $f(z)$ can have* **any and every** *complex value (except possibly one) an* **infinite** *number of times. The goal of this problem is to see how this works for a simple example.*

Problem 5: Suppose the function $f(z)$ has a singular point at $z = 0$. Let's define a neighborhood around $z = 0$ according the condition $|z| < \epsilon$, where ϵ is some positive real number. The smaller ϵ is, the smaller the neighborhood around $z = 0$. If $z = 0$ is an essential singularity, then no matter how small we take ϵ, we can find infinitely many solutions to the equation $f(z) = c$ within our neighborhood, where c is any complex number (with possibly one exception). This is not the case if $z = 0$ is a pole.

For simplicity, we will consider $c = 1$ in this problem. (The generalization to arbitrary c is left as an exercise for the curious student.)

(a) First, let's show that Picard's theorem does not hold if $f(z)$ has a pole at $z = 0$. Consider the function $f(z) = 1/z^n$, where n is a positive integer. Sketch the locations of the solutions to the equation $f(z) = 1$ in the complex plane. Argue that if ϵ is sufficiently small (in this case, smaller than 1), then no solutions to $f(z) = 1$ are enclosed within the neighborhood. **(3 points)**

(b) Now, let's suppose $f(z)$ has an essential singularity at $z = 0$. Consider the function $f(z) = e^{1/z}$. Sketch the locations of the solutions to the equation $f(z) = 1$ in the complex plane. Argue that no matter how small ϵ is, there are infinitely many solutions within the neighborhood. **(3 points)**

(c) What is the "one exception" for the function $f(z) = e^{1/z}$? That is, for what value of c is there *no solution* to the equation $f(z) = c$ within our neighborhood, for any value of ϵ? **(1 point)**

This is a nice problem, suggested by a student last year, in which you derive the Cauchy-Riemann relations in polar coordinates.

Problem 6: Consider an analytic function $f(z) = u(x, y) + iv(x, y)$. Show that if u, v are expressed in polar coordinates (r, θ), then the Cauchy-Riemann relations are **(3 points)**:

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}, \qquad \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}. \tag{2}$$