PHYS 3090: Homework 5 (due Wednesday Oct. 26)

Problem 1 (5 points): A function $f(t)$ has a Laplace transform

$$F(s) = \frac{s^2 - a^2}{(s^2 + a^2)^2} \,,$$

where a is a real, positive number. Compute $f(t)$ using the Bromwich integral.

Problem 2 (10 points):

(a) Let $f(t)$ be a function with Laplace transform $F(s)$. Prove the following: **(3 points)**

$$\int_s^\infty ds' \, F(s') = \mathcal{L}\big[f(t)/t\big] \,. \tag{1}$$

(b) Using the above result, compute $\mathcal{L}[\frac{\sin t}{t}]$. **(4 points)**

(c) Recall that the sine function is given by

$$\mathrm{Si}(a) = \int_0^a dt \, \frac{\sin t}{t} \,. \tag{2}$$

Using the results of parts (a) and (b), compute $\mathrm{Si}(\infty)$. **(3 points)**

Problem 3 (10 points): Suppose a radioactive isotope, with decay constant λ, starts leaking from a nuclear reactor at $t = 0$ with rate $R(t)$. The number of radioactive atoms present outside the leak satisfies the rate equation

$$\dot{n}(t) + \lambda \, n(t) = R(t) \,, \tag{3}$$

where λ is the decay rate. We also suppose the initial condition is $n(0) = 0$.

(a) Let's suppose that $R(t) = 0$ for $t < 0$ and that the leak "turns on" at $t = 0$. For $t \geq 0$, the leak rate is given by a periodic function $R(t) = R_0\big(1 - \cos(\omega t)\big)$, where R_0 and ω are positive, real constants. Determine $n(t)$ by taking the Laplace transform of Eq. (3). **(5 points)**

(b) Compute the Green's function $g(t, \tau)$ for the above rate equation and use this to write the solution $n(t)$ for arbitrary $R(t)$. For the rate $R(t)$ given in part (a), verify that the Green's function solution for $n(t)$ is the same as what you found in part (a). **(5 points)**