PHYS 3090: Homework 6 (due Wednesday Nov. 2)

The following problem explores Fourier transform within the context of quantum mechanics. In quantum mechanics, the wavenumber k is related to the momentum p of a particle by $p = \hbar k$.

Problem 1 (24 points): The wave function $\Psi(x,t)$ for a free quantum mechanical particle of mass m satisfies the Schrödinger equation

$$-\frac{\hbar^2}{2m}\frac{\partial^2 \Psi}{\partial x^2} = i\hbar \frac{\partial \Psi}{\partial t}\,. \tag{1}$$

$\Psi(x,t)$ is known as the *position-space* wave function since the quantity $|\Psi(x,t)|^2$ describes the probability density for finding the particle at position x at time t. The Fourier transform of the wavefunction

$$\tilde{\Psi}(k,t) = \mathcal{F}[\Psi(x,t)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dx\, e^{-ikx}\,\Psi(x,t) \tag{2}$$

is known as the *momentum-space* wave function and the quantity $|\tilde{\Psi}(k,t)|^2$ gives the probability density for finding the particle with wavenumber k (or momentum $p = \hbar k$) at time t.

Consider an initial condition $\Psi(x,0) = N\, e^{-\alpha^2 x^2/2}$, where $N = \sqrt{\alpha/\sqrt{\pi}}$, which is known as a Gaussian wave packet. The constant N is determined by the normalization condition $\int_{-\infty}^{\infty} dx |\Psi(x,0)|^2 = 1$, which says that the total probability of finding the particle at *any* position x is equal to one.

(a) Compute $\tilde{\Psi}(k,0)$. **(3 points)**

(b) The uncertainty Δx describes how spread out the wavefunction is in position. It is defined by

$$\Delta x = \sqrt{\langle x^2\rangle - \langle x\rangle^2}\,, \tag{3}$$

where $\langle x\rangle = \int_{-\infty}^{\infty} dx\, x\, |\Psi(x,t)|^2$ and $\langle x^2\rangle = \int_{-\infty}^{\infty} dx\, x^2\, |\Psi(x,t)|^2$ are the expectation values of the position and position squared, respectively. Compute Δx at time $t = 0$. **(3 points)**

Hint: It may be helpful to know $\int_{-\infty}^{\infty} dx\, x^2\, e^{-a^2x^2} = \frac{\sqrt{\pi}}{2a^3}$. This equation can be proved by taking the basic Gaussian integral formula $\int_{-\infty}^{\infty} dx\, e^{-a^2x^2} = \frac{\sqrt{\pi}}{a}$ and acting on both sides with $\frac{\partial}{\partial a}$.

(c) Similar to part (b), the uncertainty Δk describes how spread out the wavefunction is in wavenumber (or momentum). It is defined by

$$\Delta k = \sqrt{\langle k^2\rangle - \langle k\rangle^2}\,, \tag{4}$$

where $\langle k\rangle = \int_{-\infty}^{\infty} dk\, k\, |\tilde{\Psi}(k,t)|^2$ and $\langle k^2\rangle = \int_{-\infty}^{\infty} dk\, k^2\, |\tilde{\Psi}(k,t)|^2$. Compute Δk at time $t = 0$. **(3 points)**

(d) The uncertainty principle states that

$$\Delta x \Delta p \geq \frac{\hbar}{2}\,. \tag{5}$$

By multiplying your answers to parts (b) and (c), times an additional factor of $\hbar$, show that the Gaussian wavepacket at $t = 0$ saturates the lower bound of the uncertainty principle. **(1 point)**

(e) Compute $\Psi(x,t)$. Hint: Fourier transform Eq. (1) with respect to x, solve for $\tilde{\Psi}(k,t)$, and then take the inverse Fourier transform. **(5 point)**

(f) Compute $|\Psi(x,t)|^2$. Sketch what the solution looks like at time $t=0$ and at a later time t_1, as a function of x. **(3 point)**

(g) Determine the uncertainties Δx and Δk at time t. What is happening to the wave packet? **(3 points)**

(h) What is the dispersion relation for Eq. (1)? What is the wave velocity $\partial\omega/\partial k$ as a function of k? This problem illustrates that systems with a nonlinear dispersion relation (i.e., $\omega/k \neq$ constant) exhibit wave packet spreading since different Fourier modes are moving with a different wave velocity. **(3 points)**

Problem 2 (6 points): Consider the wave equation for an infinite string

$$\frac{\partial^2 y}{\partial t^2} = v^2 \frac{\partial^2 y}{\partial x^2}\,, \tag{6}$$

with the initial condition $y(x,0) = f(x)$, where $f(x)$ is an arbitrary function, and $\dot{y}(x,0) = 0$. *Using Fourier transform*, show that the solution for $t > 0$ is given by

$$y(x,t) = \tfrac{1}{2}\big(f(x-vt) + f(x+vt)\big)\,. \tag{7}$$

Hint: First, compute the Fourier transform $Y(k,t)$ in terms of $F(k)$, the Fourier transform for $f(x)$.

This shows that an initial displacement in the string separates into two oppositely-moving traveling waves that maintain the initial shape $f(x)$, each with half the amplitude.