PHYS 3090: Homework 7 (due Wednesday Nov. 30)

Problem 1 (3 points): Prove that a unitary transformation R acting on vectors $\vec{u}$ and $\vec{v}$ in a complex vector space leaves the inner product $\langle \vec{u}, \vec{v} \rangle$ invariant.

Problem 2 (15 points): Consider a double-spring system, with equal masses m and spring constants k and $\frac{3}{2}k$, where x_1 and x_2 measure the displacements from the equilibrium (see Figure 1 left).

(a) What is the potential energy V? **(3 point)**

(b) Determine the normal frequencies and the normal modes. **(3 points)**

(c) What is the general solution for $\vec{x}(t)$? **(3 points)**

(d) Suppose at $t = 0$, the system had the initial condition

$$x_1(0) = 1\,, \quad x_2 = 0\,, \quad \dot{x}_1(0) = \dot{x}_2(0) = 0\,. \tag{1}$$

Using eigenvalue methods, determine $\vec{x}(t)$ for $t > 0$. **(3 points)**

(e) Compute the linear momentum p_1 and p_2 of each of the two masses, as well as the sum $p_1 + p_2$, as a function of t. **(3 points)**

Problem 3 (20 points): Consider circular configuration of three masses shown in Figure 1 (right). All three masses (with mass m) and all three springs (with spring constant k) are fixed to move along the circle of radius r. Let the variables $(\theta_1, \theta_2, \theta_2)$ be the angular displacements of each mass from its equilibrium position.

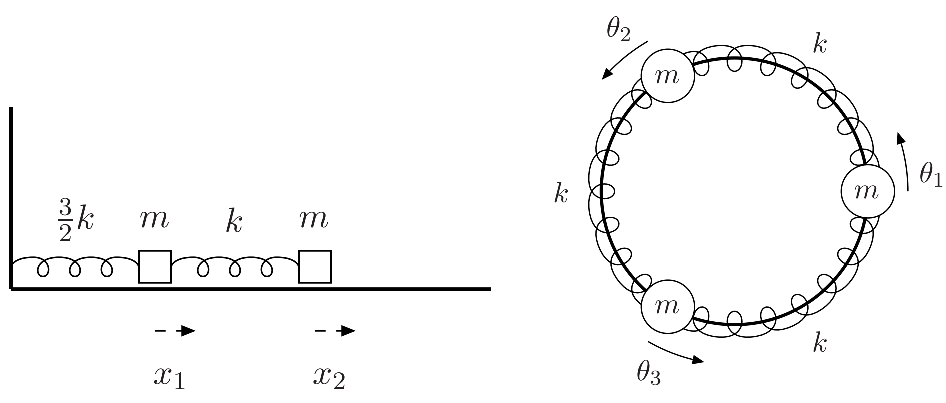

Figure 1: *Left: Double spring system with spring constants k and $\frac{3}{2}k$ and masses m on a frictionless surface (problem 2). Right: Three masses m connected by springs in a ring configuration (problem 3).*

(a) What is the potential energy V in terms of k, r, and θ_i? **(3 points)**

(b) Note that for circular motion, Newton's second law can be expressed as

$$m\ddot{\theta}_i = -\frac{1}{r^2}\frac{\partial V}{\partial \theta_i}.$$

Write the equation of motion for this system as $\ddot{\vec{\theta}} = -U\vec{\theta}$, where $\vec{\theta} = \begin{pmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{pmatrix}$ and U is a 3×3 matrix. What is U in terms of k and m? **(3 points)**

(c) What are the normal frequencies and normal modes for this system? **(3 points)**

(d) What is the general solution for $\vec{\theta}(t)$? **(3 points)**

(e) Suppose that all the masses are initially at rest in equilibrium ($\theta_i = \dot{\theta}_i = 0$) for $t < 0$. Let's generalize the equation of motion to include a driving force term

$$\ddot{\vec{\theta}} + U\vec{\theta} = \vec{F}(t)$$

where the force term is

$$\vec{F}(t) = \begin{pmatrix} \delta(t) \\ 0 \\ 0 \end{pmatrix}. \tag{2}$$

That is, mass 1 gets a "kick" at $t = 0$. Using eigenvalue methods and Laplace transform, determine $\vec{\theta}(t)$ for $t > 0$. **(5 points)**

(f) Compute the angular momentum L_1, L_2, and L_3 for each of the three masses as a function of t. Show that the total angular momentum $L_1 + L_2 + L_3$ is constant for $t > 0$. **(3 points)**

Problem 4 (10 points): The simplest type of quantum mechanical system is the **two-state system**. The unit vector $\hat{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ represents the system being in state 1 and the unit vector $\hat{e}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ represents the system being in state 2.[1] In general, the system can be in a quantum superposition of the two states, given by

$$\vec{\psi} = \psi_1\hat{e}_1 + \psi_2\hat{e}_2 = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}. \tag{3}$$

(a) Suppose at time $t = 0$ the system begins in state 1, such that $\vec{\psi}(0) = \hat{e}_1$. At time $t > 0$, the system evolves according to the Schrödinger equation

$$\frac{\partial \vec{\psi}}{\partial t} = -iH\vec{\psi}, \tag{4}$$

where

$$H = \begin{pmatrix} a & b \\ b & a \end{pmatrix} \tag{5}$$

and a, b are constants with units of 1/time. Solve Eq. (4) to determine $\vec{\psi}(t)$ using eigenvalue methods. **(7 points)**

(b) The quantities $P_i(t) = |\langle \hat{e}_1, \vec{\psi}(t)\rangle|^2$ represents the probability of finding the system in state i. Compute $P_1(t)$ and $P_2(t)$. **(3 points)**

[1] An example of a two-state system is an electron at rest, in which the two states are spin-up and spin-down.