PHYS 3090: Homework 8 (due never)

Problem 1: $SU(2)$ matrices can be represented as

$$R = \exp\left(\tfrac{i}{2}\vec{\alpha}\cdot\vec{\sigma}\right) \tag{1}$$

where $\vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ is a vector in which the components are the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{2}$$

and $\vec{\alpha} = (\alpha_1, \alpha_2, \alpha_3)$ are the parameters of the group. Let $\alpha = |\vec{\alpha}|$ and $\hat{\alpha} = \vec{\alpha}/\alpha$.

- Show that $(\hat{\alpha}\cdot\vec{\sigma})^2 = \mathbb{1}$ for any $\vec{\alpha}$, where $\mathbb{1}$ is the 2×2 identity matrix.
- Using this relation, show that
$$R = \cos\left(\tfrac{\alpha}{2}\right)\mathbb{1} + i(\hat{\alpha}\cdot\vec{\sigma})\,\sin\left(\tfrac{\alpha}{2}\right). \tag{3}$$
- Check that Eq. (3) is unitary.
- By Taylor expanding Eq. (3) to linear order in α_1, α_2 and α_3, verify that the generators are $T_i = \frac{1}{2}\sigma_i$.