Singularities are points where a function $f(z)$ is not analytic.

(1) Poles are points where $|f(z)| \to \infty$ as z approaches some value. e.g. $f(z) = \frac{1}{z-a}$ has a pole at $z=a$.

A pole at $z=a$ has order p if

$$\lim_{z\to a} (z-a)^p f(z) \quad \text{is a nonzero finite number.}$$

e.g. $f(z) = \frac{1}{z-a}$ has order $p=1$ (simple pole) since

$$\lim_{z \to a} (z-a)^n \frac{1}{z-a} = \begin{cases} 1 & \text{for } n=1 \\ 0 & \text{for } n>1 \end{cases}$$

e.g. $f(z) = \frac{1}{(z-a)^2}$ has $p=2$

e.g. $f(z) = \frac{1}{(z-a)(z-b)}$ has two simple poles (at $z=a$ and $z=b$), assuming $a \neq b$.

(2) Removable singularities are where $f(z) \to$ finite as z approaches some value.

e.g. $f(z) = \frac{\sin z}{z}$ is not analytic at $z=0$, but $\lim_{z\to 0} f(z) = 1$. So can redefine $f(z)$

as $f(z) = \begin{cases} \frac{\sin z}{z} & z \neq 0 \\ 1 & z=0 \end{cases}$ to remove the singularity

at $z=0$, and the new $f(z)$ is analytic at $z=0$.

(3) Essential singularities is a pole of infinite order and has peculiar properties as z approaches some value.

e.g. $f(z) = e^{1/z}$ has an essential singularity at $z=0$.

$= 1 + \frac{1}{z} + \frac{1}{2}\frac{1}{z^2} + \dots$ infinite number of powers of $1/z$

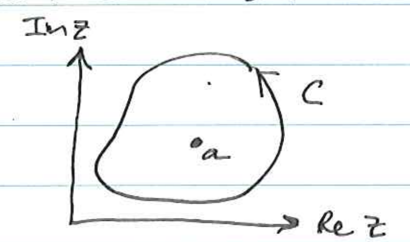
All of the above are <u>isolated</u> singularities. They occur at a point which is separated from other possible singular points in $f(z)$.

(4) <u>Branch cuts</u> (e.g. $f(z) = \ln z$) are not isolated singularities since the function is not analytic along a line.

<u>Complex integration with poles</u>

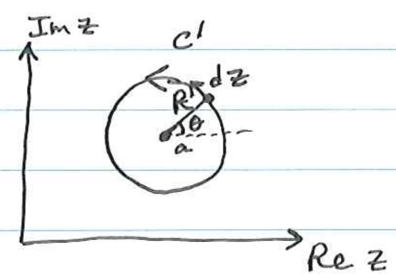
(note: orientation → counterclockwise)

Consider a function $f(z) = \dfrac{1}{z-a}$. It has a simple pole at $z=a$, ~~The integral~~ $\cancel{\oint_C f(z)}$ but is analytic for $z \neq a$. The integral $\oint_C f(z)\,dz$ is zero if C does not contain a, but nonzero if C does contain a.



We are free to deform C into a new contour C' that also contains a, and $\oint_C dz\, f(z) = \oint_{C'} dz\, f(z)$.

Choose C' to be a circle of radius R centered at $z=a$.



A point along C' can be expressed as $z = a + R(\cos\theta + i\sin\theta) = a + Re^{i\theta}$

Then $dz = \dfrac{\partial z}{\partial \theta}\, d\theta = Rie^{i\theta}\, d\theta$

To integrate over C', we want to integrate from $\theta = 0$ to $\theta = 2\pi$.

$$\oint_C dz\, f(z) = \oint_{C'} dz\, f(z) = \int_0^{2\pi} Rie^{i\theta}\, d\theta\, \frac{1}{a + Re^{i\theta} - a} = \int_0^{2\pi} i\, d\theta = 2\pi i$$

This is an example of a <u>residue</u>. Each pole in $f(z)$ within a closed path C contributes to the integral $\oint_C dz\, f(z)$, which would otherwise be zero if $f(z)$ were analytic within and on C.

Next, consider a more complicated integral $\oint_C \frac{f(z)}{(z-a)^n} dz$, where $f(z)$ is analytic on and within C.

case $n=1$: This is similar to the previous example. if $z=a$ is within C, then

$$\oint_C dz \frac{f(z)}{z-a} = \int_0^{2\pi} i\,d\theta\, f(a+Re^{i\theta})$$

Since we are free to choose any R, take $R \to 0$. Then $\oint_C dz \frac{f(z)}{z-a} = \int_0^{2\pi} i\,d\theta\, f(a) = 2\pi i\, f(a)$

case $n>1$: use a trick. Note that

$$\frac{\partial}{\partial a} \oint_C dz \frac{f(z)}{z-a} = (-1)(-1) \oint_C dz \frac{f(z)}{(z-a)^2} = \oint_C dz \frac{f(z)}{(z-a)^2}$$

$$\left(\frac{\partial}{\partial a}\right)^2 \oint_C dz \frac{f(z)}{z-a} = 2 \oint_C dz \frac{f(z)}{(z-a)^3}$$

$$\left(\frac{\partial}{\partial a}\right)^{n-1} \oint_C dz \frac{f(z)}{z-a} = (n-1)! \oint_C dz \frac{f(z)}{(z-a)^n}$$

So $$\oint_C dz \frac{f(z)}{(z-a)^n} = \frac{1}{(n-1)!}\left(\frac{\partial}{\partial a}\right)^{n-1} \oint_C dz \frac{f(z)}{z-a} = \frac{1}{(n-1)!}(2\pi i)\left(\frac{\partial}{\partial a}\right)^{n-1} f(a)$$

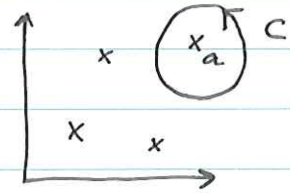
$$= \frac{2\pi i}{(n-1)!} f^{(n-1)}(a)$$

$f^{(n-1)}(z) = \frac{d^{n-1}f}{dz^{n-1}}$

Thus we have the Cauchy formula:

$$\oint_C dz \frac{f(z)}{(z-a)^n} = \begin{cases} \frac{2\pi i}{(n-1)!} f^{(n-1)}(a) & \text{if } z=a \text{ lies with } C \\ 0 & \text{otherwise} \end{cases}$$

Residues: Given a function $f(z)$ that is analytic around $z=a$, except possibly at $z=a$ itself, the residue of $f(z)$ at $z=a$ is defined by



$$\text{Res}\, f(a) = \frac{1}{2\pi i} \oint_C dz\, f(z)$$

where C encloses $z=a$ and no other singular points. If $f(z)$ is analytic at $z=a$, then $\text{Res}\, f(a) = 0$ by Cauchy's theorem.

previous example: $f(z) = \frac{1}{z-a} \rightarrow \text{Res}\, f(a) = \frac{1}{2\pi i} \times 2\pi i = 1$

previous example: $f(z) = \frac{g(z)}{(z-a)^n}$ where $g(z)$ is analytic around $z=a$.

note: $g(z) = (z-a)^n f(z)$

$$\text{Res}\, f(a) = \frac{1}{2\pi i} \oint_C dz \frac{g(z)}{(z-a)^n} = \frac{1}{2\pi i} \times 2\pi i \frac{1}{(n-1)!} g^{(n-1)}(a)$$

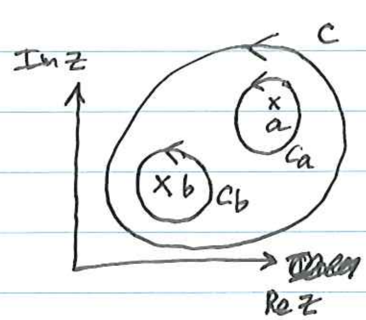
$$= \frac{1}{(n-1)!} \frac{d^{n-1}}{dz^{n-1}} \left((z-a)^n f(z) \right) \Big|_{z=a}$$

example: function with two poles, at $z=a$ and $z=b$.

$$f(z) = \frac{g(z)}{(z-a)^n (z-b)^m}$$

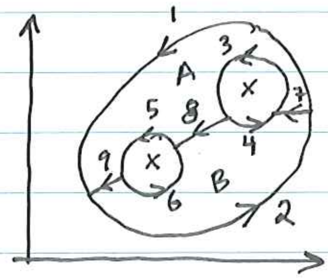
, where $a \neq b$ and $g(z)$ analytic around $z=a, b$

evaluate $\oint_C dz\, f(z)$ where C encloses both $z=a$ and $z=b$



note: $\oint_C dz\, f(z) = \oint_{C_a} dz\, f(z) + \oint_{C_b} dz\, f(z)$ where C_a and C_b each enclose only $z=a$ and $z=b$, respectively.

Can divide up C, C_a, C_b into segments; where $C = P_1 + P_2$, $C_a = P_3 + P_4$, $C_b = P_5 + P_6$



since $f(z)$ is analytic in regions A and B, can deform contours as follows:

$$\int_{P_1} dz\, f(z) = \left(\int_{P_7} + \int_{P_3} + \int_{P_8} + \int_{P_5} + \int_{P_9} \right) dz\, f(z)$$

$$\int_{P_2} dz\, f(z) = \left(\int_{-P_9} + \int_{P_6} + \int_{-P_8} + \int_{P_4} + \int_{-P_7} \right) dz\, f(z)$$

Summing them together: $\int_{P_1+P_2} dz\, f(z) = \int_{P_3+P_4} dz\, f(z) + \int_{P_5+P_6} dz\, f(z)$

$$\oint_C dz\, f(z) = \oint_{C_a} dz\, f(z) + \oint_{C_b} dz\, f(z)$$

Therefore we have: $\oint_C dz\, f(z) = 2\pi i \,\text{Res}\, f(a) + 2\pi i \,\text{Res}\, f(b)$
Straight forward to generalize to any number of poles with C.

Residue theorem:

$$\boxed{\oint_C dz\, f(z) = 2\pi i \sum_{i=1}^{N} \text{Res}\, f(a_i)}$$

where $f(z)$ is analytic on and within C, except for a finite number N of isolated singularies at $z = a_1, \ldots, a_N$ contained within C.

For a pole at $z = a$ of order n, the residue is

$$\boxed{\text{Res}\, f(a) = \lim_{z \to a} \frac{1}{(n-1)!} \frac{d^{n-1}}{dz^{n-1}} \left((z-a)^n f(z) \right)}$$

Recall: the order of a pole is obtained by finding the power p such that $\lim_{z \to a} (z-a)^p f(z) =$ a nonzero finite number.

e.g. $\frac{1}{z-a}$ has $p=1$ since $\lim_{z \to a} (z-a) \frac{1}{z-a} = 1$

e.g. $\frac{\sin z}{z^3}$ has a pole at $z=0$

$$\lim_{z \to 0} z^p \frac{\sin z}{z^3} = \lim_{z \to 0} z^p \frac{z}{z^3} = \lim_{z \to 0} z^{p-2} = \begin{cases} \infty & p=0,1 \\ 1 & p=2 \\ 0 & p>2 \end{cases}$$

$\Rightarrow$ pole of order 2

This is the main important result from complex analysis. It is very powerful: to computed integrals on closed paths, you don't need to do any actual integration! Just compute the residues (involves taking derivatives).

~~Applications of the residue theorem~~

~~The residue theorem is very useful for solving real integrals of real functions.~~

Applications of the residue theorem

To evaluate $\oint_C dz\, f(z)$:

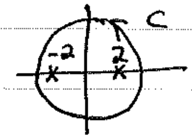
(1) find the poles of $f(z)$ and determine their orders

(2) Compute the residues for all poles enclosed within C

(3) Sum the enclosed residues (and multiply by $2\pi i$)

example: $$\oint_C dz \frac{\sin z}{z^2-4} = \oint_C dz \frac{\sin z}{(z+2)(z-2)}$$ where C defined by circle $|z|=3$

Two poles: $z=2$ and $z=-2$, both enclosed within C and have order $=1$.



$$\mathrm{Res}\, f(2) = \lim_{z\to 2} (z-2) \frac{\sin z}{(z-2)(z+2)} = \frac{\sin 2}{4}$$

$$\mathrm{Res}\, f(-2) = \lim_{z\to -2} (z+2) \frac{\sin z}{(z-2)(z+2)} = -\frac{\sin(-2)}{4} = \frac{\sin 2}{4}$$

$$\oint_C dz \frac{\sin z}{z^2-4} = 2\pi i \times 2 \times \frac{\sin 2}{4} = \pi i \sin(2)$$

example: $\oint_C dz \frac{1}{z^2 \sin z}$ where C defined by $|z|=1$ circle

pole at $z=0$. $\lim_{z\to 0} z^3 \frac{1}{z^2 \sin z} = 1 \Rightarrow$ pole has order 3.

$$\mathrm{Res}\, f(0) = \lim_{z\to 0} \frac{1}{2} \frac{d^2}{dz^2}\left(z^3 \frac{1}{z^2 \sin z}\right) = \lim_{z\to 0} \frac{1}{2} \frac{d^2}{dz^2}\left(\frac{z}{\sin z}\right)$$

Note: easy to evaluate 2nd derivative using Taylor expansion of $\sin z = z - \frac{1}{3!}z^3 + \frac{1}{5!}z^5 + \dots$

Then $$\frac{z}{\sin z} = \frac{z}{z - \frac{1}{3!}z^3 + \frac{1}{5!}z^5 + \dots} = \frac{z}{z(1-\frac{1}{3!}z^2+\frac{1}{5!}z^4 + \dots)}$$

$$\approx 1 + \frac{1}{3!}z^2 - \frac{1}{5!}z^4 + \dots$$ using $\frac{1}{1-a} \approx 1+a$ for $a \ll 1$

$$\lim_{z\to 0} \frac{d^2}{dz^2}\left(\frac{z}{\sin z}\right) = \lim_{z\to 0} \frac{d^2}{dz^2}\left(1 + \frac{1}{3!}z^2 - \frac{1}{5!}z^4 + \dots\right) = \frac{1}{3}.$$

$\mathrm{Res} = \frac{1}{6} \rightarrow \oint_C dz \frac{1}{z^2 \sin z} = \frac{1}{2}\frac{1}{3} 2\pi i = \frac{\pi i}{3}$