IV. Laplace transform

Given a function $f(t)$ defined for $0 \le t < \infty$, the Laplace transform is

$$\mathcal{L}[f(t)] = F(s) = \int_0^\infty dt\, e^{-st} f(t)$$

This is an example of an integral transform.
$F(s)$ is only defined as long as the integral converges.

ex. $f(t) = c = \text{constant}$

$$F(s) = \int_0^\infty dt\, c\, e^{-st} = -\frac{c}{s} e^{-st}\Big|_0^\infty = \frac{c}{s}$$

ex. $f(t) = t$

$$F(s) = \int_0^\infty dt\, t\, e^{-st} = -\frac{1}{s} t e^{-st}\Big|_0^\infty + \int_0^\infty dt\, \frac{1}{s} e^{-st} = \frac{1}{s^2}$$

ex. $f(t) = t^n \quad (n = 0, 1, 2, \dots)$

$$F(s) = \int_0^\infty dt\, t^n e^{-st} = \overbrace{-\frac{1}{s} t^n e^{-st}\Big|_0^\infty}^{0} + \int_0^\infty dt\, n t^{n-1} \frac{1}{s} e^{-st}$$

$$= n(n-1)(n-2)\cdots(1)\frac{1}{s^n}\int_0^\infty dt\, e^{-st} = \frac{n!}{s^{n+1}}$$

converges for $s \ge 0$, so $F(s)$ defined for $s \ge 0$.

ex. $f(t) = e^{\alpha t}$

$$F(s) = \int_0^\infty dt\, e^{(\alpha - s)t} = \frac{1}{\alpha - s} e^{(\alpha-s)t}\Big|_0^\infty = \frac{1}{s-\alpha}$$

but only converges for $s \geqslant \alpha$. So $F(s)$ defined for $s \geqslant \alpha$.

ex. $f(t) = \cos(\omega t) = \frac{1}{2}\left(e^{i\omega t} + e^{-i\omega t}\right)$

$$F(s) = \int_0^\infty dt\, \cos(\omega t)\, e^{-st}$$

$$= \frac{1}{2}\int_0^\infty dt \left(e^{(i\omega - s)t} + e^{(-i\omega - s)t}\right)$$

$$= \frac{1}{2}\left[\frac{1}{i\omega - s} e^{(i\omega - s)t} + \frac{1}{-i\omega - s} e^{(-i\omega - s)t}\right]\Bigg|_{t=0}^{\infty}$$

$$= \frac{1}{2}\left(\frac{1}{s - i\omega} + \frac{1}{s + i\omega}\right) = \frac{s + i\omega + s - i\omega}{2(s^2 + \omega^2)} = \frac{s}{s^2 + \omega^2}$$

ex. $f(t) = \sin(\omega t)$

$$F(s) = \frac{\omega}{s^2 + \omega^2}$$

$F(s)$ for $\sin(\omega t)$ & $\cos(\omega t)$ defined for $s \geq 0$.

The Laplace transform is a <u>linear operator</u>:

(1) $\mathcal{L}[c f(t)] = \int_0^\infty dt\, e^{-st} c f(t) = c \int_0^\infty dt\, e^{-st} f(t) = c\mathcal{L}[f(t)]$

(2) $\mathcal{L}[f_1(t) + f_2(t)] = \int_0^\infty dt\, e^{-st}(f_1(t) + f_2(t)) = \int_0^\infty dt\, e^{-st} f_1(t) + \int_0^\infty dt\, e^{-st} f_2(t)$

$$= \mathcal{L}[f_1(t)] + \mathcal{L}[f_2(t)]$$

Laplace transform of derivatives:

$$\mathcal{L}[f'(t)] = \int_0^\infty dt\, e^{-st} \frac{df}{dt} = e^{-st} f(t)\Big|_0^\infty + s\int_0^\infty dt\, e^{-st} f(t)$$

$$= -f(0) + sF(s)$$

$$\mathcal{L}[f''(t)] = -f'(0) + s\mathcal{L}[f'(t)]$$

$$= -f'(0) - sf(0) + s^2 F(s)$$

So an arbitrary number of derivatives gives :

$$\mathcal{L}[f^{(n)}(t)] = s^n F(s) - f^{(n-1)}(0) - s f^{(n-2)}(0) - \dots - s^{n-1} f(0)$$

Using Laplace transforms to solve differential equations :

Useful for systems that begin with an initial condition at time $t=0$ (or some other time) and then evolve for $t>0$.

ex. Simple harmonic oscillator $\ddot{x}(t) + \omega^2 x(t) = 0$

define $X(s) = \mathcal{L}[x(t)] = \int_0^\infty dt\, e^{-st} x(t)$

Take Laplace transform of diffential equation:

$$\mathcal{L}[\ddot{x}(t) + \omega^2 x(t)] = s^2 X(s) - x'(0) - s x(0) + \omega^2 X(s) = 0$$

$$\Rightarrow \quad X(s) = \frac{x'(0) + s\,x(0)}{s^2+\omega^2}$$

$$= x'(0)\frac{1}{s^2+\omega^2} + x(0)\frac{s}{s^2+\omega^2}$$

$$x(t) = \frac{x'(0)}{\omega}\sin(\omega t) + x(0)\cos(\omega t)$$

Laplace transform transforms a differential equation into an algebraic equation.

example: radioactive decay

Consider a chain of radioactive decays $1 \to 2 \to 3$, where $1,2,3$ are elements and $\lambda_{1,2,3}$ are the decay rates. ($\lambda_3 = 0 \to 3$ is stable.)

Let n_1, n_2, n_3 be the number of a given element.
Coupled equations:

$$\frac{dn_1}{dt} = -\lambda_1 n_1 \qquad \frac{dn_2}{dt} = -\lambda_2 n_2 + \lambda_1 n_1 \qquad \frac{dn_3}{dt} = +\lambda_2 n_2.$$

take Laplace transform:

$$s N_1(s) - n_1(0) = -\lambda_1 N_1(s)$$
$$s N_2(s) - n_2(0) = -\lambda_2 N_2(s) + \lambda_1 N_1(s)$$
$$s N_3(s) - n_3(0) = \lambda_2 N_2(s)$$

Solve algebraic equations:

$$N_1(s) = \frac{n_1(0)}{s+\lambda_1}$$

$$N_2(s) = \frac{n_2(0)}{s+\lambda_2} + \frac{\lambda_1 N_1(s)}{s+\lambda_2} = \frac{n_2(0)}{s+\lambda_2} + \frac{\lambda_1 n_1(0)}{(s+\lambda_1)(s+\lambda_2)}$$

$$N_3(s) = \frac{n_3(0)}{s} + \frac{\lambda_2 N_2(s)}{s}$$

$$= \frac{n_3(0)}{s} + \frac{\lambda_2}{s}\left(\frac{n_2(0)}{s+\lambda_2} + \frac{\lambda_1 n_1(0)}{(s+\lambda_1)(s+\lambda_2)} \right)$$

$$N_2(s) = \frac{n_2(0)}{s+\lambda_2} + \frac{\lambda_1 n_1(0)}{\lambda_2 - \lambda_1}\left(\frac{1}{s+\lambda_1} - \frac{1}{s+\lambda_2}\right)$$

$$= \left(n_2(0) - \frac{\lambda_1 n_1(0)}{\lambda_2 - \lambda_1}\right)\frac{1}{s+\lambda_2} + \frac{\lambda_1 n_1(0)}{\lambda_2 - \lambda_1}\frac{1}{s+\lambda_1}$$

$$N_3(s) = \frac{n_3(0)}{s} + \frac{n_2(0)}{s} - \frac{n_2(0)}{s+\lambda_2}$$

$$+ n_1(0)\left(\frac{1}{s} + \frac{\lambda_2}{\lambda_1 - \lambda_2}\frac{1}{s+\lambda_1} - \frac{\lambda_1}{\lambda_1 - \lambda_2}\frac{1}{s+\lambda_2}\right)$$

$$= \frac{1}{s}\left(n_1(0) + n_2(0) + n_3(0)\right) + n_1(0)\frac{\lambda_2}{\lambda_1 - \lambda_2}\frac{1}{s+\lambda_1}$$

$$- \left(n_2(0) + \frac{n_1(0)\lambda_1}{\lambda_1 - \lambda_2}\right)\frac{1}{s+\lambda_2}$$

Next, invert the transform:

So $n_1(t) = n_1(0)\, e^{-\lambda_1 t}$

$$n_2(t) = \left(n_2(0) + \frac{\lambda_1 n_1(0)}{\lambda_1 - \lambda_2}\right) e^{-\lambda_2 t} + \frac{\lambda_1 n_1(0)}{\lambda_2 - \lambda_1} e^{-\lambda_1 t}$$

$$n_3(t) = n_1(0) + n_2(0) + n_3(0) + n_1(0)\frac{\lambda_2}{\lambda_1 - \lambda_2} e^{-\lambda_1 t}$$

$$- \left(n_2(0) + \frac{n_1(0)\lambda_1}{\lambda_1 - \lambda_2}\right) e^{-\lambda_2 t}$$

$$n_2(t) = n_2(0)\, e^{-\lambda_2 t} + n_1(0)\frac{\lambda_1}{\lambda_2 - \lambda_1}\left(e^{-\lambda_1 t} - e^{-\lambda_2 t}\right)$$

$$n_3(t) = n_3(0) + n_2(0)\left(1 - e^{-\lambda_2 t}\right)$$

$$+ n_1(0)\left(1 + \frac{\lambda_2}{\lambda_1 - \lambda_2} e^{-\lambda_1 t} - \frac{\lambda_1}{\lambda_1 - \lambda_2} e^{-\lambda_2 t}\right)$$

Inverse Laplace transform

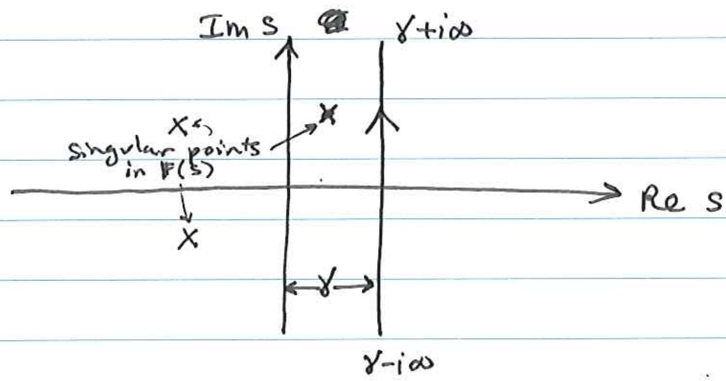
Easy way to invert the Laplace transform is by inspection (i.e. reducing $F(s)$ to a form where you can guess $f(t)$)

More general procedure: Bromwich integral

Given $F(s)$, the inverse Laplace transformed function $f(t)$ is

$$f(t) = \frac{1}{2\pi i}\int_{\gamma - i\infty}^{\gamma + i\infty} ds\, e^{st}\, F(s)$$

where γ is any real number chosen so that the contour lies to the right of any ~~poles~~ singularities in $F(s)$.



proof: take the Laplace transform

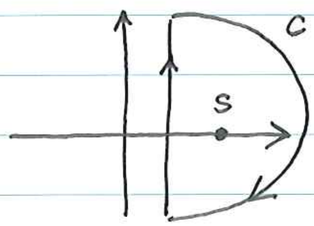
$$\int_0^\infty dt\, f(t)\, e^{-st} = \int_0^\infty dt\, e^{-st} \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} ds'\, e^{s't}\, F(s')$$

$$= \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} ds'\, F(s') \int_0^\infty dt\, e^{-(s-s')t}$$

↑ if $s > \gamma$, then integral converges

$$= \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} ds'\, \frac{-F(s')}{s'-s}$$

Now close the contour in $Re(s) > 0$ half-plane:



$$\int_{\gamma-i\infty}^{\gamma+i\infty} ds' = \oint_C ds'$$

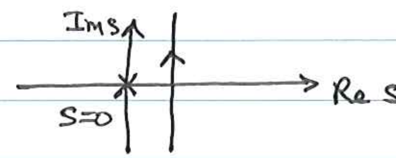
provided $F(s') \to 0$ for $s' \to \infty$.

$$\Rightarrow \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} ds' \, \frac{F(s')}{s-s'} = \frac{1}{2\pi i}\oint_C ds' \, \frac{-F(s')}{s'-s}$$

$$= \frac{1}{2\pi i}(-2\pi i)(-F(s)) = F(s)$$

example: $F(s) = \frac{1}{s^2}$. pole at $s=0 \to$ choose $\gamma > 0$.

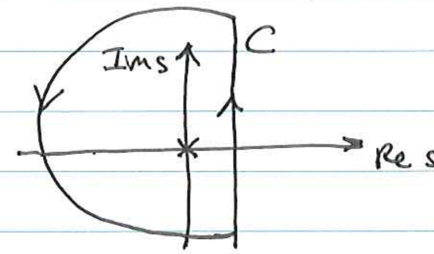
$$f(t) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} ds \, e^{st} \frac{1}{s^2}$$



Note: closing the contour at $Re(s) \to \infty$ will not converge. Need to close the contour at $Re(s) \to -\infty$.

$$f(t) = \frac{1}{2\pi i}\oint_C ds \, e^{st} \frac{1}{s^2}$$

where

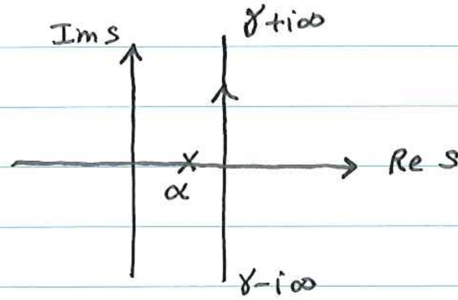


$$f(t) = \frac{1}{2\pi i} \times 2\pi i \; \mathrm{Res}\left[e^{st}\frac{1}{s^2} \, ; \, s=0\right]$$

$$= \lim_{s\to 0} \frac{d}{ds}\left(s^2 \frac{e^{st}}{s^2}\right) = \lim_{s\to 0} t e^{st} = t$$

so $f(t) = t$.

example: $F(s) = \dfrac{1}{s-\alpha}$ (where $\alpha > 0$ is a real number)



Must choose $\gamma > \alpha$.

$$f(t) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} ds\, e^{st}\,\frac{1}{s-\alpha} = \frac{1}{2\pi i}\oint_C ds\,\frac{e^{st}}{s-\alpha}$$

close contour at $Re(s) \to -\infty$.

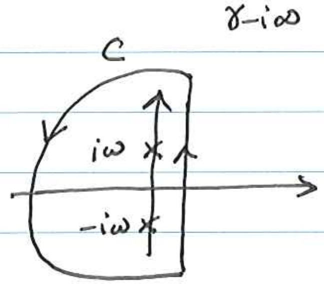
pole at $s = \alpha$.

$$f(t) = 2\pi i\,\frac{1}{2\pi i}\,Res\left[\frac{e^{st}}{s-\alpha}\,;\; s=\alpha\right] = e^{\alpha t}$$

example: $F(s) = \dfrac{s}{s^2+\omega^2} = \dfrac{s}{(s+i\omega)(s-i\omega)}$ ($\omega > 0$, real)

choose $\gamma > Re(\pm i\omega) = 0$

$$f(t) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} ds\, e^{st}\,\frac{s}{(s+i\omega)(s-i\omega)} = \frac{1}{2\pi i}\oint_C ds\,\frac{s\,e^{st}}{(s+i\omega)(s-i\omega)}$$



$$= Res\left[\frac{s e^{st}}{(s+i\omega)(s-i\omega)}\,;\; s=i\omega\right]$$

$$+ Res\left[\frac{s\,e^{st}}{s^2+\omega^2}\,;\; s=-i\omega\right]$$

$$= \frac{i\omega\, e^{i\omega t}}{2i\omega} + \frac{i\omega\, e^{-i\omega t}}{-2i\omega} = \cos(\omega t)$$

Dirac delta function

Defined by $\delta(x-a) = \begin{cases} \infty & x = a \\ 0 & x \neq a \end{cases}$

and $\int_{-\infty}^{\infty} dx\, \delta(x-a) = 1.$

Some useful properties:

δ-function is even: $\delta(x-a) = \delta(a-x)$

$$\delta(c(x-a)) = \frac{1}{|c|}\delta(x-a)$$

$$\int_{-\infty}^{\infty} dx\, f(x)\,\delta(x-a) = f(a)$$

The δ-function is the derivative of the step function:

$$\theta(x-a) = \begin{cases} 1 & x > a \\ 0 & x < a \end{cases}$$

$$= \int_{-\infty}^{x} dx'\, \delta(x'-a)$$

$$\frac{d}{dx}\theta(x-a) = \delta(x-a)$$

Laplace transform:

let $f(t) = \delta(t-\tau)$

then $F(s) = \int_0^\infty dt\, e^{-st}\,\delta(t-\tau) = e^{-s\tau}$.

Green's functions

Very powerful technique for solving inhomogeneous, linear differential equations. Example: forced simple harmonic oscillator

$$\ddot{x} + 2\gamma \dot{x} + \omega^2 x = F(t)$$

linear = each term on the left side only involves power of $x(t)$ or its derivative

inhomogeneous = forcing function $F(t) \neq 0$.

Consider the case where the system is at rest for $t < 0$:

i.e. $x(t) = \dot{x}(t) = F(t) = 0$ for $t < 0$.

Then $F(t)$ "turns on" at $t = 0$. Solve for $x(t)$ with initial condition $x(0) = \dot{x}(0) = 0$.

Trick: first consider a simpler equation

$$\ddot{g} + 2\gamma \dot{g} + \omega^2 g = \delta(t - \tau) \qquad (\text{where } \tau > 0)$$

The solution to this equation $g(t, \tau)$ is a Green's function. After solving for $g(t, \tau)$, the solution for $x(t)$ is:

$$x(t) = \int_0^\infty d\tau \, g(t, \tau) \, F(\tau)$$

Proof: plug this solution into the original equation:

$$\left(\frac{d^2}{dt^2} + 2\gamma \frac{d}{dt} + \omega^2\right) x(t) = \int_0^\infty d\tau \left(\frac{d^2}{dt^2} + 2\gamma \frac{d}{dt} + \omega^2\right) g(t, \tau) \, F(\tau)$$

$$= \int_0^\infty d\tau \, \delta(t - \tau) \, F(\tau) = F(t)$$

The Green's function describes the response of the system to an impulse at $t=\tau$.

The total response $x(t)$ to an arbitrary driving force $F(t)$ is the "sum" (integral) over all infinitesimal impulses for $0<\tau<\infty$.

Now, solve for $g(t,\tau)$ using Laplace transform:

$$G(s,\tau) = \int_0^\infty dt\, e^{-st}\, g(t,\tau)$$

So we have, acting with $\int_0^\infty dt\, e^{-st}$ on both sides:

$$\left(s^2 G(s,\tau) - s\, g(0,\tau) - \dot{g}(0,\tau)\right) + 2\gamma\left(s G(s,\tau) - g(0,\tau)\right)$$

$$+\ \omega^2 G(s,\tau) = e^{-s\tau}$$

Take $g(t,\tau)$ to obey same initial conditions as $x(t)$.
So $g(0,\tau) = \dot{g}(0,\tau) = 0$.

$$(s^2 + 2\gamma s + \omega^2)\, G(s,\tau) = e^{-s\tau}$$

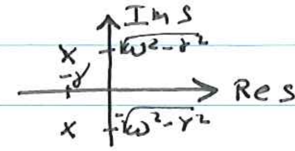
$$G(s,\tau) = \frac{e^{-s\tau}}{s^2+2\gamma s+\omega^2} = \frac{e^{-s\tau}}{(s-\lambda_+)(s-\lambda_-)}$$

where $\lambda_\pm = \frac{1}{2}\left(-2\gamma \pm \sqrt{4\gamma^2 - 4\omega^2}\right) = -\gamma \pm \sqrt{\gamma^2-\omega^2}$

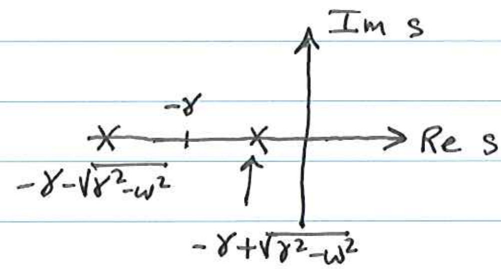
$G(s,\tau)$ has simple poles at $s = \lambda_+$ and $s = \lambda_-$.

(i) case $\omega > \gamma \rightarrow \pm\sqrt{\gamma^2 - \omega^2} = \pm i\sqrt{\omega^2 - \gamma^2}$ is imaginary

$\rightarrow$ poles at $s = -\gamma \pm i\sqrt{\omega^2 - \gamma^2}$



(ii) case $\omega < \gamma \rightarrow \pm\sqrt{\gamma^2 - \omega^2}$ is real



poles are always in the $\mathrm{Re}(s) < 0$ plane. Can take Bromwich contour $\int_{-i\infty}^{i\infty} ds$.

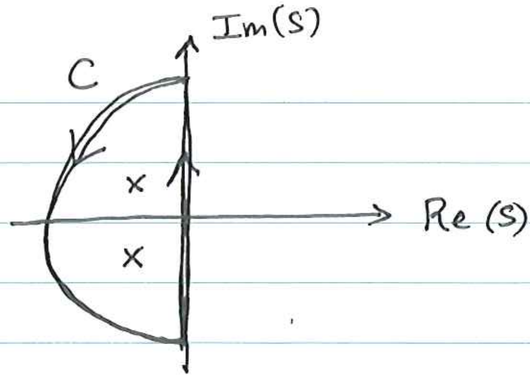
$$g(t,\tau) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} ds\, e^{st}\, G(s,\tau)$$

$$= \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} ds \frac{e^{(t-\tau)s}}{(s-\lambda_+)(s-\lambda_-)}$$

How do we want to close the contour? Want the integrand to go to zero when we close the contour.

case 1: $t > \tau$. $e^{(t-\tau)s} \rightarrow 0$ for $\mathrm{Re}(s) \rightarrow -\infty$
$e^{(t-\tau)s} \rightarrow \infty$ for $\mathrm{Re}(s) \rightarrow +\infty$

close the contour in the $\mathrm{Re}(s) < 0$ plane.



$$g(t,\tau) = \frac{1}{2\pi i}\, 2\pi i \left\{ \mathrm{Res}(\lambda_+) + \mathrm{Res}(\lambda_-) \right\}$$

$$= \lim_{s \to \lambda_+} (s-\lambda_+) \frac{e^{(t-\tau)s}}{(s-\lambda_+)(s-\lambda_-)} + (\lambda_+ \leftrightarrow \lambda_-)$$

$$= \frac{e^{(t-\tau)\lambda_+}}{\lambda_+ - \lambda_-} + \frac{e^{(t-\tau)\lambda_-}}{\lambda_- - \lambda_+}$$

$$= \frac{1}{2\sqrt{\gamma^2-\omega^2}} \left[e^{(t-\tau)\lambda_+} - e^{(t-\tau)\lambda_-} \right]$$

Let's work in the limit $\gamma \ll \omega$ for simplicity:

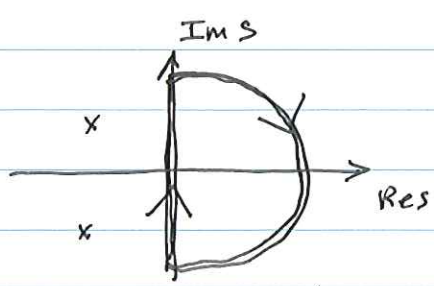
$$g(t,\tau) = \frac{1}{2i\omega} \left[e^{-\gamma(t-\tau)+i\omega(t-\tau)} - e^{-\gamma(t-\tau)-i\omega(t-\tau)} \right]$$

$$= \frac{1}{\omega} e^{-\gamma(t-\tau)} \sin\left(\omega(t-\tau)\right)$$

case 2: $t < \tau$. $\quad e^{(t-\tau)s} \to \infty$ for $\mathrm{Re}(s) \to -\infty$

$e^{(t-\tau)s} \to 0$ for $\mathrm{Re}(s) \to +\infty$

close contour in the $\mathrm{Re}(s) > 0$ plane



No enclosed poles.
So $g(t,\tau) = 0$

So we have: $g(t,\tau) = \begin{cases} 0 & t<\tau \\ \frac{1}{\omega} e^{-\gamma(t-\tau)} \sin(\omega(t-\tau)) & t>\tau \end{cases}$

$$= \theta(t-\tau)\frac{1}{\omega} e^{-\gamma(t-\tau)} \sin(\omega(t-\tau))$$

Note: 1. response only depends on $t-\tau$ (time elapsed after the impulse), not the absolute time t.

2. $g(t,\tau)=0$ for $t<\tau \rightarrow$ causality. There is no response before the impulse occurs.

For any forcing function $F(t)$, the solution for $x(t)$ is:

$$x(t) = \int_0^\infty d\tau \frac{1}{\omega}\theta(t-\tau) e^{-\gamma(t-\tau)} \sin(\omega(t-\tau)) F(\tau)$$

$$= \int_0^t d\tau \frac{1}{\omega} e^{-\gamma(t-\tau)} \sin(\omega(t-\tau)) F(\tau)$$

Solution $x(t)$ only depends on times $\tau<t$, not on times $\tau>t$ (causality)