Group theory

A group G is a set of objects $g_1, g_2, \cdots$, plus a group operation $\circ$, such that the following are satisfied:

(1) closure: if g_1, g_2 are in G, then $g_1 \circ g_2$ is in G.

(2) associativity: $g_1 \circ (g_2 \circ g_3) = (g_1 \circ g_2) \circ g_3$

(3) identity: there exist an identity element e in G such that for all g_i in G

$$g_i \circ e = e \circ g_i = g_i$$

(4) inverse: every g_i has an inverse g_i^{-1} in G such that

$$g_i \circ g_i^{-1} = g_i^{-1} \circ g_i = e.$$

Note: group operation is not necessarily commutative. $g_1 \circ g_2 \neq g_2 \circ g_1$ in general.
Some groups are commutive, in that case $g_1 \circ g_2 = g_2 \circ g_1$.

Examples of Groups:

- G = integers $(0, \pm 1, \pm 2, \dots)$ under addition ($\circ = +$)

  Discrete, infinite group.

  Sum of any two integers is another integer.

- $G =$ integers $0, 1, 2, \cdots, N-1$ under modular addition.
Integers modulo N. $n \bmod N =$ remainder of n/N.

Hour hand on a clock ($N=12$). $G = 0(=12), 1, 2, \cdots, 11$

$$10+7 = 17 \bmod 12 = 5$$
$$6+6 = 12 \bmod 12 = 0$$
$$1+3 = 4 \bmod 12 = 4 \qquad \text{etc.}$$

This is a discrete finite group.

- $G =$ vectors in $\mathbb{R}^n$ (n-dim space) under addition.

$\vec{v} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}$. Continuous (infinite) group.
Parametrized by continuous parameters
$v_1, v_2, \cdots =$ any real number

closure: $\vec{v} + \vec{u} = \begin{pmatrix} v_1 + u_1 \\ \vdots \end{pmatrix}$ is a vector in G.

- $G =$ group of rotations in 2-dim. under matrix multiplication.

group element $g(\alpha) = R(\alpha) = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$

to satisfy closure $g(\alpha_1) \circ g(\alpha_2)$ must be in G.

$$g(\alpha_1) \circ g(\alpha_2) = R(\alpha_1) R(\alpha_2) = \begin{pmatrix} \cos\alpha_1 & -\sin\alpha_1 \\ \sin\alpha_1 & \cos\alpha_1 \end{pmatrix} \begin{pmatrix} \cos\alpha_2 & -\sin\alpha_2 \\ \sin\alpha_2 & \cos\alpha_2 \end{pmatrix}$$

$$= \begin{pmatrix} \cos\alpha_1 \cos\alpha_2 - \sin\alpha_1 \sin\alpha_2 & -\cos\alpha_1 \sin\alpha_2 - \sin\alpha_1 \cos\alpha_2 \\ \sin\alpha_1 \cos\alpha_2 + \cos\alpha_1 \sin\alpha_2 & \cos\alpha_1 \cos\alpha_2 - \sin\alpha_1 \sin\alpha_2 \end{pmatrix}$$

$$= \begin{pmatrix} \cos(\alpha_1+\alpha_2) & -\sin(\alpha_1+\alpha_2) \\ \sin(\alpha_1+\alpha_2) & \cos(\alpha_1+\alpha_2) \end{pmatrix} = R(\alpha_1+\alpha_2) = g(\alpha_1+\alpha_2)$$

Combination of any two rotations is a rotation.
Group is commutative: $R(\alpha_1)\,R(\alpha_2) = R(\alpha_2)\,R(\alpha_1) = R(\alpha_1+\alpha_2)$

Identity: $e = R(0) = \mathbb{1}$

Inverse: $g^{-1}(\alpha) = R(-\alpha)$

We're going to focus on continuous groups, also known as <u>Lie Groups</u>. Each group element is parametrized by a set of real ~~parameters~~ numbers.

Most (but not all) Lie Groups can be represented by square matrices & group operation is matrix multiplication.

<u>example</u>: vectors $\vec{V}$ in $\mathbb{R}^n$ under addition.

For starters, consider $n=1$, i.e. $G=$ real numbers under addition. How can this be represented as matrix multiplication?

Any two real numbers u, v under the group operation gives $u+v =$ another real number.

We want to find a representation D such that for each group element v, there is a matrix $D(v)$. The matrices must satisfy: ~~the conditions~~

Since $u \circ v = u + v$, then $D(u)D(v) = D(u+v)$.

Let $D(v) = \begin{pmatrix} 1 & v \\ 0 & 1 \end{pmatrix}$ be a represention of v.
Then

$$D(u)D(v) = \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & v \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & u+v \\ 0 & 1 \end{pmatrix} = D(u+v)$$

works!

For arbitrary n, $\vec{v} = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$. Representation:

$$D(\vec{v}) = \begin{pmatrix} \begin{matrix} 1 & v_1 \\ 0 & 1 \end{matrix} & 0 & & 0 \\ 0 & \begin{matrix} 1 & v_2 \\ 0 & 1 \end{matrix} & 0 & \\ & 0 & \ddots & 0 \\ 0 & & 0 & \begin{matrix} 1 & v_n \\ 0 & 1 \end{matrix} \end{pmatrix}$$

$$D(\vec{u})D(\vec{v}) = D(\vec{u}+\vec{v})$$

So addition of vectors can be represented by matrix multiplication.

Exponentiation of matrices

Useful to consider e^M where M is a square matrix.
What does it mean? Taylor expand:

$$e^M = \mathbb{1} + M + \frac{1}{2!}M^2 + \frac{1}{3!}M^3 + \ldots$$

where $M^n = \underbrace{MM\cdots M}_{n \text{ times}}$ matrix multiplication

Example: Let $M = \begin{pmatrix} 0 & -\alpha \\ \alpha & 0 \end{pmatrix}$. Define $T = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$.
Then $M = i\alpha T$.

Compute $e^M = \mathbb{1} + M + \frac{1}{2!} M^2 + \dots$

$$= \mathbb{1} + i\alpha T + \frac{1}{2!}(i\alpha T)^2 + \dots$$

$$= \mathbb{1} + i\alpha T + \frac{1}{2!}(i\alpha)^2 T^2 + \dots$$

$$T^2 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} = \mathbb{1}$$
$$T^3 = T^2 T = \mathbb{1} T = T$$
$$T^4 = T^2 T^2 = \mathbb{1}\,\mathbb{1} = \mathbb{1} \quad \text{etc.}$$

$T^n = \mathbb{1}$ for $n =$ even
$T^n = T$ for $n =$ odd

$$e^M = \left(1 + \frac{1}{2!}(i\alpha)^2 + \frac{1}{4!}(i\alpha)^4 + \dots\right)\mathbb{1}$$
$$+ \left(i\alpha + \frac{1}{3!}(i\alpha)^3 + \frac{1}{5!}(i\alpha)^5 + \dots\right) T$$
$$= \left(1 - \frac{1}{2!}\alpha^2 + \frac{1}{4!}\alpha^4 + \dots\right)\mathbb{1}$$
$$+ i\left(\alpha - \frac{1}{3!}\alpha^3 + \frac{1}{5!}\alpha^5 + \dots\right) T$$
$$= \cos\alpha\, \mathbb{1} + i \sin\alpha\, T = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} = R(\alpha)$$

Rotation can be ~~expes~~ expressed in exponential form as $R(\alpha) = \exp(i\alpha T)$, where

$T = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$ is the <u>generator</u> of the group.

<u>Example:</u> Compute e^M for $M = \begin{pmatrix} 0 & v \\ 0 & 0 \end{pmatrix}$.

Let $T = \begin{pmatrix} 0 & -i \\ 0 & 0 \end{pmatrix}$, then $M = ivT$.

Note: $M^2 = \begin{pmatrix} 0 & v \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & v \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = 0$

so all higher powers of M vanish as well.

$$e^M = \mathbb{1} + M = \begin{pmatrix} 1 & v \\ 0 & 1 \end{pmatrix} = e^{ivT} = D(v)$$

This is the representation of real numbers under addition.
$T = \begin{pmatrix} 0 & -i \\ 0 & 0 \end{pmatrix}$ is the generator.

Note: convention (among physicists) to factor out the i explicitly (but not among mathematicians)

A Lie group has one generator $T_1, T_2, \ldots, T_n$ for each parameter $\alpha_1, \alpha_2, \ldots, \alpha_n$ parametrizing the group element.

$$g = \exp\left(i \sum_{i=1}^{n} T_i \alpha_i\right)$$

n = dimension of the group = # of parameters = # of generators.

<u>Finding the generators</u>

Exponentiation allows you to construct the group elements from the generators.

If you know the group element, you can find the generators by taking the infinitesimal limit.

For 2-d rotations:

If $\alpha \ll 1$, Taylor expand $R(\alpha) = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} = \begin{pmatrix} 1 & -\alpha \\ \alpha & 1 \end{pmatrix}$

at linear order in α.

Also $R(\alpha) = \exp(i\alpha T) = \mathbb{1} + i\alpha T = \mathbb{1} + \begin{pmatrix} 0 & -\alpha \\ \alpha & 0 \end{pmatrix}$

$$= \mathbb{1} + i\alpha \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \Rightarrow T = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$$

For real numbers: $D(v) = \exp(ivT) = \mathbb{1} + v\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

$$= \mathbb{1} + ivT \Rightarrow T = \begin{pmatrix} 0 & -i \\ 0 & 0 \end{pmatrix}.$$

Lie groups & Lie algebras

Some important Lie groups

$O(N)$ = orthogonal group: $N \times N$ orthogonal matrices R $(R^{-1} = R^T)$

$SO(N)$ = special orthogonal group: $N \times N$ orthogonal matrices R such that $\det(R) = 1$.

$U(N)$ = unitary group: $N \times N$ unitary matrices R $(R^\dagger = R^{-1})$

$SU(N)$ = special unitary group: $N \times N$ unitary matrices R such that $\det(R) = 1$.

$SU(N)$ is a <u>subgroup</u> of $U(N)$ and $SO(N)$ is a subgroup of $U(N)$.

Subgroups are themselves groups and are subsets of some larger group.

Lie Groups can be connected or disconnected.

<u>Connected group</u>: Every element of G is related to every other by a series of infinitesimal transformations. Continuity among elements of the group.

example: $SO(2)$ = rotation matrices in 2-dim $R(\alpha) = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$.

Note: $\det R(\alpha) = \cos^2\alpha + \sin^2\alpha = 1$.

Infinitesimal rotation: $R(\delta\alpha) \cong \begin{pmatrix} 1 & -\delta\alpha \\ \delta\alpha & 1 \end{pmatrix} = \mathbb{1} + i\,\delta\alpha\, T$

where $\delta\alpha \ll 1$.

Start with identity element $e = \mathbb{1}$. Any element $R(\alpha)$ can be expressed as a combination of infinitesimal rotations acting on e.

Let $\delta\alpha = \frac{\alpha}{N}$. $\quad R(\alpha) = \lim_{N\to\infty} R(\frac{\alpha}{N})\ e = \lim_{N\to\infty} (\mathbb{1} + i\frac{\alpha}{N}T)^N$

$$= e^{i\alpha T}$$

Recall: $\quad e^x = \lim_{N\to\infty} (1 + \frac{x}{N})^N$.

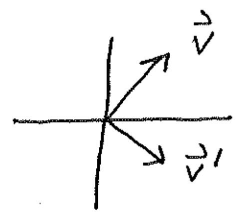
All elements of a simply connected group can be expressed in exponential form.

<u>Disconnected group</u>: Not all elements of G are connected to one another by continuous transformations.

example: $O(2)$ = group of rotations & reflections in 2-Dim.

Reflection about x-axis:

$$\vec{v} = \begin{pmatrix} v_x \\ v_y \end{pmatrix} \to \vec{v}' = \begin{pmatrix} v_x \\ -v_y \end{pmatrix} = O\,\vec{v}$$



where $\quad O = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

Note: reflection about any axis can be expressed as combination of rotations and matrix O.

e.g. reflection about y-axis:

$$R(-\pi/2)\ O\ R(\pi/2) = \begin{pmatrix} 0 & +1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

also $\quad O^2 = OO = \mathbb{1}$

$\qquad\quad O R(\alpha) = R(-\alpha)\,O$

So arbitrary combination of rotations & reflections:

$$R(\alpha_1)\,O\,R(\alpha_2)\,O \ldots O\,R(\alpha_n) = \begin{cases} R(\alpha_1)R(-\alpha_2)\ldots R(-\alpha_n) & \text{even } O\text{'s} \\ R(\alpha_1)R(-\alpha_2)\ldots R(-\alpha_n)\,O & \text{odd } O\text{'s} \end{cases}$$

$$= \begin{cases} R(\alpha_1 - \alpha_2 - \ldots - \alpha_n) & \text{even \# reflections} \to \text{rotation} \\ R(\alpha_1 - \alpha_2 - \ldots - \alpha_n)\,O & \text{odd \# reflections} \to \text{rotation + one reflection.} \end{cases}$$

$$= \begin{cases} R(\alpha) \\ R(\alpha)\,O \end{cases}$$

even # of reflections : $\det(R(\alpha)) = 1$
odd # of reflections : $\det(R(\alpha)O) = \det(R(\alpha))\det(O) = +1 \cdot (-1) = -1$

there is no infinitesimal transformation that can produce a reflection. Group elements with det $= +1$ and -1 are disconnected. Note: elements with det $= -1$ do not form a subgroup since doesn't contain identity and not closed.

Two groups G, G' are <u>isomorphic</u> if there is a one-to-one correspondence between its elements $g_i \leftrightarrow g_i'$ which preserves the group operation:

i.e. if $g_1 \circ g_2 = g_3$ then $g_1' \circ g_2' = g_3'$

example: $U(1)$ = group of unitary 1×1 "matrices"
i.e. complex phases $e^{i\alpha}$.

Under a $U(1)$ transformation: complex number $v = v_x + i v_y$

$$\rightarrow v' = e^{i\alpha} v = (\cos\alpha + i\sin\alpha)(v_x + i v_y)$$

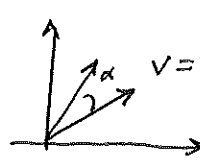
$$= \underbrace{(\cos\alpha\, v_x - \sin\alpha\, v_y)}_{v_x'} + i\underbrace{(\sin\alpha\, v_x + \cos\alpha\, v_y)}_{v_y'}$$

Same as $SO(2)$ transformation:

$$\vec{v}' = R(\alpha)\vec{v} = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}\begin{pmatrix} v_x \\ v_y \end{pmatrix}$$

Also $R(\alpha_1)R(\alpha_2) = R(\alpha_1 + \alpha_2)$, $e^{i\alpha_1}e^{i\alpha_2} = e^{i(\alpha_1+\alpha_2)}$

$SO(2)$ and $U(1)$ are isomorphic.



$v = v_x + i v_y$, $\vec{v} = \begin{pmatrix} v_x \\ v_y \end{pmatrix}$

Lie groups can be:

<u>Abelian</u>: Group is commutative, i.e. $g_1 \circ g_2 = g_2 \circ g_1$ for <u>any</u> two group elements.

<u>Nonabelian:</u> Group is not commutative

So far we have only considered abelian groups.

<u>example:</u> the simplest nonabelian group (and of major importance in physics) is $SU(2)$.

$SU(2)$ = 2x2 unitary matrices with det = 1.

$$R = \begin{pmatrix} a & -b^* \\ b & a^* \end{pmatrix}$$ parametrize $SU(2)$ matrix in terms of two complex parameters a, b.

$$\det R = |a|^2 + |b|^2 = 1 \Rightarrow$$ additional constraint on a, b.

. R parametrized in terms of 3 real parameters.
$\Rightarrow$ dimension of $SU(2)$ is 3 $\Rightarrow$ 3 generators, T_1, T_2, T_3.

unitarity: $$R^\dagger R = \begin{pmatrix} a^* & b^* \\ -b & a \end{pmatrix}\begin{pmatrix} a & -b^* \\ b & a^* \end{pmatrix} = \begin{pmatrix} |a|^2+|b|^2 & 0 \\ 0 & |a|^2+|b|^2 \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbb{1}.$$

Find generators of $SU(2)$: Want to find T_i such that

$$R = \exp\left(i \sum_{i=1}^{3} \alpha_i T_i\right).$$ Show T_i are 2x2 Hermitian, traceless matrices.

We can pick α_i to be anything. Take only $i=j$ nonzero and $i \neq j$ zero. Work in limit that $\alpha_j \ll 1$ (infinitesimal transformation)

$$\Rightarrow R = \mathbb{1} + i \alpha_j T_j$$

Consider $R^\dagger R = \mathbb{1}$ at linear order in α_j:

$$\mathbb{1} = R^\dagger R \doteq (\mathbb{1} - i\alpha_j T_j^\dagger)(\mathbb{1} + i\alpha_j T_j) = \mathbb{1} + i\alpha_j (T_j - T_j^\dagger)$$

$\Rightarrow \quad T_j = T_j^\dagger \qquad$ Generators must be Hermitian.

Also require $\det R = 1$.

$$R = \mathbb{1} + i\alpha_j T_j = \begin{pmatrix} 1 + i\alpha_j (T_j)_{11} & i\alpha_j (T_j)_{12} \\ i\alpha_j (T_j)_{21} & 1 + i\alpha_j (T_j)_{22} \end{pmatrix}$$

$$\det R = 1 + i\alpha_j (T_j)_{11} + i\alpha_j (T_j)_{22} + \ldots \quad \text{order } \alpha_j^2$$

$$= 1 + i\alpha_j \,\mathrm{Tr}[T_j] = 1$$

$$\Rightarrow \quad \mathrm{Tr}[T_j] = 0 .$$

You may already be familiar with a set of 2×2 Hermitian, traceless matrices: Pauli matrices.

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

It is convention to define the $SU(2)$ generators with an additional factor of $1/2$. Standard convention: $\mathrm{Tr}[T_i T_j] = \frac{1}{2}\delta_{ij}$.

$$T_1 = \tfrac{1}{2}\sigma_1 = \tfrac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad T_2 = \tfrac{1}{2}\sigma_2 = \tfrac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad T_3 = \tfrac{1}{2}\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\tfrac{1}{2}$$

Let's exponentiate $\exp\left(i \sum_{i=1}^{3} \alpha_i T_i\right)$ to show it takes the form $\begin{pmatrix} a & -b^* \\ b & a^* \end{pmatrix}$.

Define vector $\vec{\alpha} = (\alpha_1, \alpha_2, \alpha_3)$, $\vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$.
Also define $\alpha = |\vec{\alpha}|$ and $\hat{\alpha} = \vec{\alpha}/\alpha$.

Then $\exp\left(i \sum_{i=1}^{3} \alpha_i T_i \right) = \exp\left(\frac{i}{2} \vec{\alpha}\cdot\vec{\sigma} \right)$

Can show that: (HW)

$$\exp\left(\frac{i}{2} \vec{\alpha}\cdot\vec{\sigma} \right) = \cos\left(\frac{\alpha}{2}\right) \mathbb{1} + i(\hat{\alpha}\cdot\vec{\sigma}) \sin\left(\frac{\alpha}{2}\right).$$

$\hat{\alpha}$ is a unit vector. Parametrize it as

$$\hat{\alpha} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$$

$$\hat{\alpha}\cdot\vec{\sigma} = \begin{pmatrix} \cos\theta & \sin\theta\, e^{-i\phi} \\ \sin\theta\, e^{+i\phi} & -\cos\theta \end{pmatrix}$$

$$R = \exp\left(\frac{i}{2} \vec{\alpha}\cdot\vec{\sigma} \right) = \begin{pmatrix} \cos\left(\frac{\alpha}{2}\right) + i\cos\theta\sin\left(\frac{\alpha}{2}\right) & i\sin\theta\, e^{-i\phi}\sin\left(\frac{\alpha}{2}\right) \\ i\sin\theta\, e^{i\phi}\sin\left(\frac{\alpha}{2}\right) & \cos\left(\frac{\alpha}{2}\right) - i\cos\theta\sin\left(\frac{\alpha}{2}\right) \end{pmatrix}$$

$$= \begin{pmatrix} a & -b^* \\ b & a^* \end{pmatrix}$$

$$a = \cos\left(\frac{\alpha}{2}\right) + i\cos\theta\sin\left(\frac{\alpha}{2}\right)$$
$$b = i\sin\theta\, e^{i\phi}\sin\left(\frac{\alpha}{2}\right)$$

Generators T_i of a Lie group form a vector space known as a <u>Lie algebra</u>.

i.e. $\sum_{i=1}^{n} \alpha_i T_i$ is a vector: T_i are basis vectors, α_i are the components.

Plus the generators satisfy commutation relations (Lie bracket)

$$[T_i, T_j] = T_i T_j - T_j T_i = \sum_{k=1}^{n} i f_{ijk} T_k$$

where f_{ijk} are <u>structure constants</u>.

For an abelian group, $f_{ijk} = 0$.

Proof: Closure property of a group requires that any combination of group elements is itself an element of the group.

Define $R(\vec{\alpha}) = \exp\left(i \sum_{i=1}^{n} \alpha_i T_i\right)$.

For an abelian group, $R(\vec{\alpha}) R(\vec{\beta}) = R(\vec{\alpha} + \vec{\beta})$, but not for a nonabelian group. Let's consider the following object:

$$R(\vec{\alpha}) R(\vec{\beta}) R(-\vec{\alpha} - \vec{\beta}) = \exp\left(i \sum_{i=1}^{n} \alpha_i T_i\right) \exp\left(i \sum_{i=1}^{n} \beta_i T_i\right)$$

$$\times \exp\left(-i \sum_{i=1}^{n} (\alpha_i + \beta_i) T_i\right)$$

By closure, there must be some $\vec{\gamma}$ such that

$$R(\vec{\alpha}) R(\vec{\beta}) R(-\vec{\alpha} - \vec{\beta}) = R(\vec{\gamma}).$$

Let $\vec{\alpha} = (0, \ldots, \alpha_i, \ldots 0)$ only i-component is non zero

$\vec{\beta} = (0, \ldots, \beta_j, \ldots, 0)$ only j-component is non-zero

Treat $\alpha_i, \beta_j \ll 1$ as infinitesimal & expand to 2nd order.

$$R(\vec{\alpha}) R(\vec{\beta}) R(-\vec{\alpha}-\vec{\beta}) = \left(\mathbb{1} + i\alpha_i T_i - \tfrac{1}{2}(\alpha_i T_i)^2\right)\left(\mathbb{1} + i\beta_j T_j - \tfrac{1}{2}(\beta_j T_j)^2\right)$$

$$\times \left(\mathbb{1} - i(\alpha_i T_i + \beta_j T_j) - \tfrac{1}{2}(\alpha_i T_i + \beta_j T_j)^2\right)$$

$$= \mathbb{1} + i\cancel{\alpha_i T_i} + \cancel{i\beta_j T_j} - i\left(\cancel{\alpha_i T_i + \beta_j T_j}\right)$$

$$- \tfrac{1}{2}\Big(\cancel{(\alpha_i T_i)^2} + \cancel{(\beta_j T_j)^2} + \cancel{2}\alpha_i\beta_j T_i T_j - \cancel{2\alpha_i^2 T_i^2} - \cancel{2(\alpha_i T_i)(\beta_j T_j)}$$

$$- \cancel{2}(\beta_j T_j)(\alpha_i T_i) - 2\cancel{(\beta_j T_j)^2} + \cancel{(\alpha_i T_i)^2} + \cancel{(\beta_j T_j)^2}$$

$$+ \cancel{(\alpha_i T_i)(\beta_j T_j)} + \cancel{(\beta_j T_j)(\alpha_i T_i)} \Big)$$

$$= \mathbb{1} - \tfrac{1}{2}\alpha_i\beta_j [T_i, T_j]$$

$$= \mathbb{1} + i\sum_{k=1}^{n} \gamma_k T_k = R(\vec{\gamma})$$

We need $\displaystyle\sum_{k=1}^{n} \gamma_k T_k = \frac{i}{2}\alpha_i\beta_j [T_i, T_j]$

$$\mathrm{Tr}\left[T_m \sum_{k=1}^{n} \gamma_k T_k\right] = \tfrac{1}{2}\gamma_m = \tfrac{i}{2}\alpha_i\beta_j\, \mathrm{Tr}\left[T_m [T_i, T_j]\right]$$

$$\gamma_k = i\,\alpha_i\beta_j \underbrace{\mathrm{Tr}\left[[T_i, T_j] T_k\right]}_{\text{structure constant } f_{ijk}}$$

For SU(2): For Pauli matrices $[\sigma_i, \sigma_j] = 2i\,\varepsilon_{ijk}\,\sigma_k$

So $f_{ijk} = \varepsilon_{ijk}$ = Levi-Civita antisymmetric tensor

$$= \begin{cases} 0 & \text{if any } i,j,k \text{ are equal} \\ \pm 1 & \text{for any even } (+1) \text{ or odd } (-1) \text{ permutation of } i=1,\ j=2,\ k=3. \end{cases}$$

e.g. $\varepsilon_{123} = \varepsilon_{231} = \varepsilon_{312} = -\varepsilon_{132} = -\varepsilon_{321} = -\varepsilon_{213} = +1$

$\varepsilon_{111} = \varepsilon_{112} = \varepsilon_{113} = \varepsilon_{122} = \varepsilon_{133} = \ldots \text{etc.} = 0$

Representations

Rep. D satisfies $D(g_1) \circ D(g_2) = D(g_3)$ if $g_1 \circ g_2 = g_3$.
Preserves group operation.

We're going to take SU(2) as a concrete example:

$$g_i = R(\vec{\alpha}) = \exp\left(i \sum_{i=1}^{3} \alpha_i T_i\right)$$

Fundamental rep: $D(R) = R$
Antifundamental rep: $D(R) = R^*$
Trivial rep: $D(R) = \mathbb{1}$

Before we get to more complicated reps, introduce some nomenclature:

SU(2) matrices R act on two-component vectors $\vec{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$

$$R\vec{v} \quad \text{or} \quad \sum_j R_{ij} V_j \quad \text{in components.}$$

Refer to $\vec{v}$ as "a fundamental" – i.e. transforms under fundamental representation. $V_i \to R_{ij} V_j$.

From now on: use summation convention that repeated indices are summed over. $R_{ij} V_j = \sum_{j=1}^{2} R_{ij} V_j$

Then $\vec{u}^*$ is an antifundamental: $u_i^* \to R_{ij}^* u_j$

<u>Bifundamental representation:</u> Consider an object with two indices M_{ij}, transforming as

$$M_{ij} \to M_{ij}' = R_{ik} R_{jl} M_{kl}$$

in matrix form $M \to M' = R M R^T$, $M = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix}$

Define vector $\vec{m} = \begin{pmatrix} M_{11} \\ M_{21} \\ M_{12} \\ M_{22} \end{pmatrix}$. Want to find rep $D(R)$ such that $M \to M' = R M R^T$ corresponds to $\vec{m} \to \vec{m}' = D(R)\,\vec{m}$ where $D(R)$ is 4x4 matrix.

Write: $R = \begin{pmatrix} a & -b^* \\ b & a^* \end{pmatrix}$ Then:

$$M' = R M R^T = \begin{pmatrix} a & -b^* \\ b & a^* \end{pmatrix} \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix}$$

$$= \begin{pmatrix} aM_{11} - b^* M_{21} & aM_{12} - b^* M_{22} \\ bM_{11} + a^* M_{21} & bM_{12} + a^* M_{22} \end{pmatrix} \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix}$$

$$= \left(\begin{array}{c|c} a^2 M_{11} - ab^* M_{21} - ab^* M_{12} + b^{*2} M_{22} & ab M_{11} - |b|^2 M_{21} + |a|^2 M_{12} - a^* b^* M_{22} \\ \hline ab M_{11} + |a|^2 M_{21} - |b|^2 M_{12} - a^* b^* M_{22} & b^2 M_{11} + a^* b M_{21} + a^* b M_{12} + a^{*2} M_{22} \end{array}\right)$$

$$\vec{M} \to \vec{M}' = \begin{pmatrix} a^2 & -ab^* & -ab^* & b^{*2} \\ ab & |a|^2 & -|b|^2 & -a^*b^* \\ ab & -|b|^2 & |a|^2 & -a^*b^* \\ b^2 & a^*b & a^*b & a^{*2} \end{pmatrix} \begin{pmatrix} M_{11} \\ M_{21} \\ M_{12} \\ M_{22} \end{pmatrix}$$

$$= \left(\begin{array}{c|c} aR & -b^* R \\ \hline bR & a^* R \end{array}\right) \begin{pmatrix} M_{11} \\ M_{21} \\ M_{12} \\ M_{22} \end{pmatrix} = D(R)\,\vec{m}$$

So $D(R) = \underbrace{\begin{pmatrix} aR & -b^*R \\ bR & a^*R \end{pmatrix}}_{4\times4}$, $\quad R = \begin{pmatrix} a & -b^* \\ b & a^* \end{pmatrix}$

Check: $D(R)$ unitary: $D(R)^\dagger D(R) = \begin{pmatrix} a^*R^\dagger & b^*R^\dagger \\ -b R^\dagger & aR \end{pmatrix} \begin{pmatrix} aR & -b^*R \\ bR & a^*R \end{pmatrix}$

$$= \begin{pmatrix} (|a|^2+|b|^2)R^\dagger R & 0 \\ 0 & (|a|^2+|b|^2)R^\dagger R \end{pmatrix}$$

$$= \begin{pmatrix} \mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix} = 4\times4 \text{ identity matrix.}$$

A representation is <u>reducible</u> if there exists a similarity transformation S that can reduce $D(R)$ to block diagonal form for all $D(R)$.

$$D(R) \to S^{-1} D(R) S = \begin{pmatrix} D^{(1)}(R) & & \\ & D^{(2)}(R) & \\ & & \ddots \end{pmatrix}$$

Then:
$D^{(1)}(R)$, ... etc. are irreducible representations of R. (Assuming they cannot be reduced further themselves)

example: bifundamental rep. can be reduced.

Note how quantity $(M_{12} - M_{21})$ transforms:

$$
\begin{aligned}
M_{12} - M_{21} \to M'_{12} - M'_{21} &= ab\, M_{11} - |b|^2 M_{21} + |a|^2 M_{12} - a^* b^* M_{22} \\
&\quad - ab\, M_{11} - |a|^2 M_{21} + |b|^2 M_{12} + a^* b^* M_{22} \\
&= (|a|^2 + |b|^2)(M_{12} - M_{21}) \\
&= (M_{12} - M_{21})
\end{aligned}
$$

This quantity transforms under the trivial rep. (i.e. it is invariant)

Define a new vector $\vec{m}_S = \begin{pmatrix} (M_{12} - M_{21})/\sqrt{2} \\ M_{11} \\ (M_{12} + M_{21})/\sqrt{2} \\ M_{22} \end{pmatrix}$

Then can define $S = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1/\sqrt{2} & 0 & 1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$, $S^{-1} = \begin{pmatrix} 0 & -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

$\vec{m}_S$ is related to $\vec{m}$ by $\vec{m} = S \vec{m}_S$ or $\vec{m}_S = S^{-1} \vec{m}$.

$$
S^{-1} D(R) S = \left(\begin{array}{c|ccc} 1 & 0 & 0 & 0 \\ \hline 0 & a^2 & -\sqrt{2}\, ab^* & b^{*2} \\ 0 & \sqrt{2}\, ab & |a|^2 - |b|^2 & -\sqrt{2}\, a^* b^* \\ 0 & b^2 & \sqrt{2}\, a^* b & a^{*2} \end{array} \right)
$$

Transformation of $\vec{m}_S$ is:

$$\vec{m}_S \rightarrow \vec{m}_S' = S^{-1}\vec{m}' = S^{-1}D(R)\,S\,S^{-1}\vec{m} = \underbrace{S^{-1}D(R)\,S}_{\text{block diagonal.}}\ \vec{m}_S$$

$\dfrac{M_{12}-M_{21}}{\sqrt{2}}$ transforms under trivial rep.
(Singlet) $D(R) = 1$

$\begin{pmatrix} M_{11} \\ \dfrac{M_{12}+M_{21}}{\sqrt{2}} \\ M_{22} \end{pmatrix}$ transforms under 3×3 rep. (triplet)

$$D(R) = \begin{pmatrix} a^2 & -\sqrt{2}\,a b^* & b^{*2} \\ \sqrt{2}\,ab & |a|^2-|b|^2 & -\sqrt{2}\,a^* b^* \\ b^2 & \sqrt{2}\,a^* b & a^{*2} \end{pmatrix}$$

Singlet (trivial) and triplet representations are irreducible reps.

Generators: recall $R = \begin{pmatrix} a & -b^* \\ b & a^* \end{pmatrix} = \mathbb{1} + \dfrac{i}{2}\underbrace{\begin{pmatrix} \alpha_3 & \alpha_1 - i\alpha_2 \\ \alpha_1 + i\alpha_2 & -\alpha_3 \end{pmatrix}}_{(\vec{\alpha}\cdot\vec{\sigma})}$
in infinitesimal limit

$$a = 1 + \frac{i}{2}\alpha_3\,, \quad b = \alpha_1 + i\alpha_2$$

Write $D(R) = \mathbb{1} + \displaystyle\sum_{i=1}^{3} \alpha_i T_i$

$$= \begin{pmatrix} 1+i\alpha_3 & -\sqrt{2}(\alpha_1 - i\alpha_2) & 0 \\ \sqrt{2}(\alpha_1+i\alpha_2) & 1 & -\sqrt{2}(\alpha_1 - i\alpha_2) \\ 0 & \sqrt{2}(\alpha_1 + i\alpha_2) & 1 - i\alpha_3 \end{pmatrix}$$

$$T_1 = -i\sqrt{2}\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad T_2 = \sqrt{2}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$T_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$