Laurent expansion

If a function, $f(z)$, is analytic within a given region centered at $z=a$ (including at $z=a$ itself), then $f(z)$ can be expanded as a power series in $(z-a)$. This is the usual Taylor expansion.

$$f(z) = \sum_{n=0}^{\infty} c_n (z-a)^n \quad , \text{ where } \quad c_n = \frac{1}{n!} f^{(n)}(a)$$

$$= f(a) + f'(a)(z-a) + \frac{1}{2!} f''(a)(z-a)^2 + \ldots$$

If $f(z)$ has a singularity at $z=a$, then it cannot be Taylor expanded about $z=a$. However, it can written as a Laurent expansion:

$$f(z) = \sum_{n=-p}^{\infty} c_n (z-a)^n$$

i.e. similar to a Taylor expansion but including both positive and negative powers of $(z-a)$.

If $f(z)$ has a pole of order p at $z=a$, then $\frac{1}{(z-a)^p}$ is the most negative power of $(z-a)$ appearing in the series.

~~Example~~

How to find the Laurent series: suppose $f(z)$ has a pole of order p at $z=a$. Then $f(z)$ can be written as

$$f(z) = \frac{g(z)}{(z-a)^p} \quad \text{where } g(z) \text{ is analytic at } z=a.$$

$g(z) = (z-a)^p f(z)$ can be Taylor expanded about $z=a$.

$$\Rightarrow \quad g(z) = g(a) + g'(a)(z-a) + \ldots$$

Then the Laurent series is $f(z) = \frac{g(a)}{(z-a)^p} + \frac{g'(a)}{(z-a)^{p-1}} + \frac{1}{2}\frac{g''(a)}{(z-a)^{p-2}} + \ldots$

~~Example~~

Given a Laurent expansion for $f(z) = \sum_{n=-p}^{\infty} c_n (z-a)^n$ about $z=a$, the residue of $f(z)$ at $z=a$ is $\boxed{\text{Res } f(a) = c_{-1}}$.

proof:
$$\text{Res } f(a) = \lim_{z \to a} \frac{1}{(p-1)!} \frac{d^{p-1}}{dz^{p-1}} \left((z-a)^p \sum_{n=-p}^{\infty} c_n (z-a)^n \right)$$

$$= \lim_{z \to a} \frac{1}{(p-1)!} \frac{d^{p-1}}{dz^{p-1}} \left(c_{-p} + c_{-p+1}(z-a) + \dots + c_{-1}(z-a)^{p-1} + c_0 (z-a)^p + c_1 (z-a)^{p+1} + \dots \right)$$

$$= \lim_{z \to a} \frac{1}{(p-1)!} \left(c_{-1} \cdot (p-1)! + c_0\, p!\,(z-a) + c_1 (p+1)! (z-a)^2 + \dots \right)$$

$$= c_{-1}$$

example: $f(z) = \dfrac{\sin z}{z^2-4}$, expand around $z=2$.

$$f(z) = \frac{\sin z}{(z-2)(z+2)} = \frac{1}{z-2} \frac{\sin z}{z+2}$$

$$= \frac{1}{z-2} \left(\frac{\sin 2}{4} + \left(\frac{\cos 2}{4} - \frac{\sin 2}{16} \right)(z-2) + \dots \right)$$

$$c_{-1} = \frac{\sin 2}{4} = \text{Res } f(2)$$

example: $f(z) = \dfrac{1}{z^2 \sin z} = \dfrac{1}{z^3} \left(1 + \dfrac{1}{3!} z^2 - \dfrac{1}{5!} z^4 + \dots \right)$

pole at $z=0$.

$$= \frac{1}{z^3} + \frac{1}{6}\frac{1}{z} - \frac{1}{5!} z + \dots \Rightarrow \text{Res } f(0) = \frac{1}{6}$$

Alternative method of computing the residue of $f(z)$: expand $f(z)$ in a Laurent series and obtain the coefficient c_{-1}.

example: $f(z) = \sin(1/z) = \dfrac{1}{z} - \dfrac{1}{3!}\dfrac{1}{z^3} + \dfrac{1}{5!}\dfrac{1}{z^5} + \dots$

$f(z)$ has an essential singularity at $z=0$.

$\text{Res } f(0) = 1$ since $c_{-1} = 1$.

~~VIII~~ Applications of the residue theorem

The residue theorem is very useful for solving real integrals.

Contour integration: relate real integral to a complex integral along a contour in the complex plane.

Integrals of trigonometric functions

Consider an integral of the form $\int_0^{2\pi} d\theta\, I(\cos\theta, \sin\theta)$ where I is a function of $\cos\theta, \sin\theta$. This integral can be written as an integral in the complex plane along the unit circle defined by $|z| = 1$. ~~written as~~

Use the following relations: $z = re^{i\theta} = e^{i\theta}$ $(r=1)$

$$dz = ie^{i\theta}d\theta = iz\,d\theta \Rightarrow d\theta = \frac{dz}{iz}$$

$$\cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta}) = \frac{1}{2}\left(z + \frac{1}{z}\right)$$

$$\sin\theta = \frac{1}{2i}(e^{i\theta} - e^{-i\theta}) = \frac{1}{2i}\left(z - \frac{1}{z}\right)$$

Then $\int_0^{2\pi} d\theta\, I(\cos\theta, \sin\theta) = \oint_C \frac{dz}{iz} I\left(\frac{1}{2}\left(z+\frac{1}{z}\right), \frac{1}{2i}\left(z-\frac{1}{z}\right)\right)$

where C is the unit circle $(|z|=1)$.

example: $\int_0^{2\pi} d\theta \frac{1}{2+\sin\theta} = \oint_C \frac{dz}{iz} \frac{1}{2+\frac{1}{2i}\left(z-\frac{1}{z}\right)} = \oint_C dz \underbrace{\frac{2}{z^2+4iz-1}}_{f(z) = \frac{2}{z^2+4iz-1}}$

find the poles where $z^2 + 4iz - 1 = 0$. Use quadratic formula:

$$z = \frac{1}{2}\left(-4i \pm \sqrt{(4i)^2 - 4(-1)}\right) = (-2 \pm \sqrt{3})i = z_\pm$$

only $z_+ = -2 + \sqrt{3}i$ falls with the unit circle. (Simple pole)

~~$\text{Res} f((-2+\sqrt{3})i) = \lim_{z\to(-2+\sqrt{3})i}$~~

$$\text{Res} f(z_+) = \lim_{z\to z_+} (z - z_+) \frac{2}{(z-z_+)(z-z_-)} = \frac{2}{z_+ - z_-} = \frac{2}{2\sqrt{3}i} = \frac{1}{\sqrt{3}i}$$

$$\int_0^{2\pi} d\theta \frac{1}{2+\sin\theta} = 2\pi i \frac{1}{\sqrt{3}i} = \frac{2\pi}{\sqrt{3}}$$

<u>example:</u> $\int_0^{\pi} d\theta \, \frac{1}{1+\cos^2\theta}$, Not the full unit circle since $0<\theta<\pi$.

note: $\cos(2\pi-\theta) = \cos\theta \Rightarrow \int_{\pi}^{2\pi} d\theta \, \frac{1}{1+\cos^2\theta} = \int_0^{\pi} d\theta \, \frac{1}{1+\cos^2\theta}$

$$\text{so } \int_0^{\pi} d\theta \, \frac{1}{1+\cos^2\theta} = \frac{1}{2}\int_0^{2\pi} d\theta \, \frac{1}{1+\cos^2\theta}$$

$$= \frac{1}{2}\oint_C \frac{dz}{iz}\, \frac{1}{1+\frac{1}{4}(z+1/z)^2} \cancel{= \frac{1}{2}\oint_C dz}$$

$$= \frac{1}{2i}\oint_C dz \, \frac{4z}{4z^2+z^4+2z^2+1} = \frac{1}{i}\oint dz \, \frac{2z}{z^4+6z^2+1}$$

find the poles: $z^4+6z^2+1=0$

$$z^2 = \frac{1}{2}\left(-6 \pm \sqrt{36-4}\right) = -3 \pm 2\sqrt{2}$$

note: $-3+2\sqrt{2} < 0$

roots:

$$\left.\begin{array}{l} z_1 = i\sqrt{3-2\sqrt{2}} \\ z_2 = -i\sqrt{3-2\sqrt{2}} \end{array}\right\} \text{within } C$$

$$\left.\begin{array}{l} z_3 = i\sqrt{3+2\sqrt{2}} \\ z_4 = -i\sqrt{3+2\sqrt{2}} \end{array}\right\} \text{not within } C$$

let $f(z) = \frac{2z}{z^4+6z^2+1} = \frac{2z}{(z-z_1)(z-z_2)(z-z_3)(z-z_4)}$

$$\text{Res } f(z_1) = \frac{2z_1}{(z_1-z_2)(z_1-z_3)(z_1-z_4)} \qquad \text{Res } f(z_2) = \frac{2z_2}{(z_2-z_1)(z_2-z_3)(z_2-z_4)}$$

$$\text{Res } f(z_1) + \text{Res } f(z_2) = \frac{2}{z_1-z_2}\left[\frac{z_1}{(z_1-z_3)(z_1-z_4)} - \frac{z_2}{(z_2-z_3)(z_2-z_4)}\right]$$

$$= \frac{2}{2z_1}\left[\frac{z_1}{(z_1-z_3)(z_1+z_3)} - \frac{-z_1}{(-z_1-z_3)(-z_1+z_3)}\right]$$

$$= \left[\frac{1}{(z_1^2-z_3^2)} + \frac{1}{(z_1^2-z_3^2)}\right] = \frac{2}{(z_1^2-z_3^2)}$$

$$= \frac{2}{-(3-2\sqrt{2})+(3+2\sqrt{2})} = +\frac{2}{4\sqrt{2}} = \frac{1}{2\sqrt{2}}$$

$$\int_0^{\pi} d\theta \, \frac{1}{1+\cos^2\theta} = \frac{\pi}{\sqrt{2}}$$

Integrals along the real axis

Residue theorem useful for evaluating integrals of the form $\int_{-\infty}^{\infty} dx\, f(x)$.

The principal value of an integral is

$$\text{p.v.} \int_{-\infty}^{\infty} dx\, f(x) = \lim_{R\to\infty} \int_{-R}^{R} dx\, f(x)$$

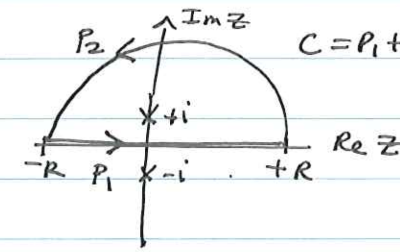
Note that the P.V. may be well-defined even if the integral $\int_{-\infty}^{\infty} dx\, f(x)$ is not.
For example, take $f(x) = x$:

$$\text{p.v.} \int_{-\infty}^{\infty} dx\, f(x) = \lim_{R\to\infty} \int_{-R}^{R} dx\, x = \lim_{R\to\infty} \tfrac{1}{2}\left(R^2 - (-R)^2\right) = 0$$

$$\text{but} \int_{-\infty}^{\infty} dx\, x = \int_{0}^{\infty} dx\, x + \int_{-\infty}^{0} dx\, x = \infty - (-\infty) \text{ does not exist since not well-defined}$$

But if $\int_{-\infty}^{\infty} dx\, f(x)$ does exist, then it is equal to the P.V.: $\int_{-\infty}^{\infty} dx\, f(x) = \text{p.v.} \int_{-\infty}^{\infty} dx\, f(x)$.

Closing the contour: P.V. $\int_{-\infty}^{\infty} dx\, f(x)$ corresponds to a path P_1 in the complex plane from $z = -R$ to $z = +R$. We want to ~~add~~ close this into a loop by adding another path P_2 from $z = +R$ back to $z = -R$ along which the integral is known.



example: $$\text{P.V.} \int_{-\infty}^{\infty} dx\, \frac{1}{x^2+1} = \lim_{R\to\infty} \int_{P_1} dz\, \frac{1}{z^2+1}$$

$$= \oint_C dz\, \frac{1}{z^2+1} - \lim_{R\to\infty} \int_{P_2} dz\, \frac{1}{z^2+1}$$

P_2 is a semicircle at $|z| = R$.

Therefore $$\int_{P_2} dz\, \frac{1}{z^2+1} = \int_0^{\pi} d\theta\, Re^{i\theta} i \frac{1}{R^2 e^{2i\theta} + 1}$$ using $z = Re^{i\theta}$, $dz = Re^{i\theta} i\, d\theta$

$$\lim_{R\to\infty} \int_{P_2} dz\, \frac{1}{z^2+1} = \lim_{R\to\infty} \int_0^{\pi} d\theta\, Re^{i\theta} i \frac{e^{-2i\theta}}{R^2} \frac{1}{\left(1 + e^{-2i\theta}\frac{1}{R^2}\right)}$$

$$\lim_{R\to\infty}\int_{P_2} dz\,\frac{1}{z^2+1} = \lim_{R\to\infty}\frac{1}{R}\int_0^{\pi} d\theta\; i e^{-i\theta}\left(1-\frac{1}{R^2}e^{-2i\theta}+\frac{1}{R^4}e^{-4i\theta}+\dots\right)$$

by Taylor expanding $\frac{1}{(1+e^{-2i\theta}/R^2)}$ in powers of $1/R^2$

$$= \lim_{R\to\infty}\frac{1}{R}\left\{2-\frac{2}{3}\frac{1}{R^2}+\frac{2}{5}\frac{1}{R^4}+\dots\right\} = 0$$

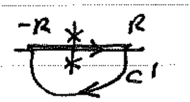
Therefore $\text{p.v.}\int_{-\infty}^{\infty} dx\,\frac{1}{x^2+1} = \oint_C dz\,\frac{1}{z^2+1}$

$f(z)=\frac{1}{z^2+1}$ has simple poles at $z=\pm i$. only $z=+i$ is within C.

$$\text{Res}\, f(i) = \lim_{z\to i}(z-i)\frac{1}{(z-i)(z+i)} = \frac{1}{2i}$$

$$\text{p.v.}\int_{-\infty}^{\infty} dx\,\frac{1}{x^2+1} = 2\pi i \times \frac{1}{2i} = \pi$$

Same result if we closed the contour as a semicircle below the real axis.



Pick up pole at $z=-i$.

Clockwise contour has extra $(-)$.

$$\text{p.v.}\int_{-\infty}^{\infty} dx\,\frac{1}{x^2+1} = \oint_{C'} dz\,\frac{1}{z^2+1} = -2\pi i\;\text{Res}\, f(-i)$$

$$= -2\pi i \lim_{z\to -i}(z+i)\frac{1}{(z-i)(z+i)} = \pi$$

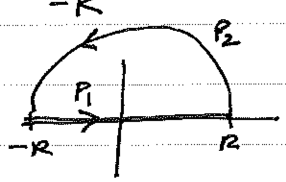
Since the integral is well-defined in this case, $\int_{-\infty}^{\infty} dx\,\frac{1}{x^2+1} = \pi$.

<u>example</u>: compute $\int_{-\infty}^{\infty} dx\,\frac{\cos x}{x^2+1}$

Since $\cos x = \text{Re}(e^{ix})$, we can write

$$\text{p.v.}\int_{-\infty}^{\infty} dx\,\frac{\cos x}{x^2+1} = \text{Re}\left[\text{p.v.}\int_{-\infty}^{\infty} dx\,\frac{e^{ix}}{x^2+1}\right] = \lim_{R\to\infty}\text{Re}\int_{-R}^{R} dx\,\frac{e^{ix}}{x^2+1}$$

$$= \lim_{R\to\infty}\text{Re}\int_{P_1} dz\,\frac{e^{iz}}{z^2+1}$$



$C = P_1 + P_2$

Again, close the contour by adding another path P_2 at $|z|=R$ in the upper half-plane.

$$\int_{P_2} dz \frac{e^{iz}}{z^2+1} = \int_0^{\pi} \underbrace{d\theta\, iRe^{i\theta}}_{dz}\ \underbrace{e^{iR\cos\theta - R\sin\theta}}_{e^{iz}}\ \underbrace{\frac{1}{(R^2 e^{i2\theta}+1)}}_{\frac{1}{z^2+1}}$$

$$= \lim_{R\to\infty} \int_0^{\pi} d\theta \frac{1}{R}\, ie^{-i\theta}\left(1 - \frac{1}{R^2}e^{-2i\theta} + \frac{1}{R^4}e^{-4i\theta} + \dots\right) e^{iR\cos\theta - R\sin\theta}$$

Note the factor of $e^{-R\sin\theta} \to 0$ for $R\to\infty$ since $\sin\theta > 0$ for $0 < \theta < \pi$. $\Rightarrow \int_{P_2} dz \frac{e^{iz}}{z^2+1} = 0$

If we had closed the contour in the negative imaginary ~~plane~~ half-plane, we would have $\int_0^{-\pi} d\theta \Rightarrow e^{-R\sin\theta} \to \infty$ since $\sin\theta > 0$. So we are required to close the contour in the positive imaginary half-plane.

$$\int_{-\infty}^{\infty} dx \frac{\cos x}{x^2+1} = \mathrm{Re} \oint_C dz \frac{e^{iz}}{z^2+1} = \mathrm{Re}\left\{2\pi i \lim_{z\to i} (z-i) \frac{e^{iz}}{(z-i)(z+i)}\right\}$$

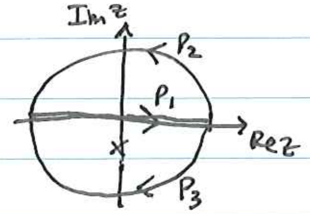
$$= \mathrm{Re}\left\{2\pi i \frac{e^{i\cdot i}}{2i}\right\} = \mathrm{Re}\left[e^{-1}\pi\right] = \frac{\pi}{e}$$

example: $\int_{-\infty}^{\infty} dx \frac{\sin x}{x+i}$. Note: $\int_{-\infty}^{\infty} dx \frac{\sin x}{x+i} \neq \mathrm{Im}\int_{-\infty}^{\infty} dx \frac{e^{ix}}{x+i}$.

$$\text{p.v.}\int_{-\infty}^{\infty} dx \frac{\sin x}{x+i} = \text{P.V.}\int_{-\infty}^{\infty} dx \frac{e^{ix}-e^{-ix}}{2i(x+i)}$$

Expand $\sin x$ and compute contributions separately.

$$= \text{P.V.}\int_{-\infty}^{\infty} dx \frac{e^{ix}}{2i(x+i)} - \text{P.V.}\int_{-\infty}^{\infty} dx \frac{e^{-ix}}{2i(x+i)}$$



$$= \underbrace{\oint_C dz \frac{e^{iz}}{2i(z+i)}}_{\text{close above}} - \underbrace{\oint_{C'} dz \frac{e^{-iz}}{2i(z+i)}}_{\text{close below}}$$

$C = P_1 + P_2$
$C' = P_1 + P_3$

$$= 0 - (-1)\, 2\pi i \frac{1}{2i} e^{-i(-i)} = \frac{\pi}{e}$$

Poles along the real axis

The integral $\int_{-\infty}^{\infty} dx\, f(x)$ is not defined if $f(x)$ has a pole on the real axis. If it has a pole at $x=a$, we define the principal value

$$\text{p.v.} \int_{-\infty}^{\infty} dx\, f(x) = \lim_{\substack{R\to\infty \\ \varepsilon\to 0}} \int_{-R}^{a-\varepsilon} dx\, f(x) + \int_{a+\varepsilon}^{R} dx\, f(x)$$

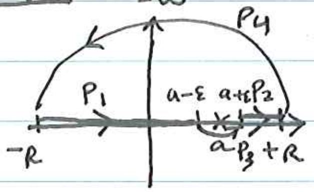
Again, where $\int_{-\infty}^{\infty} dx\, f(x)$ does exist, $\int_{-\infty}^{\infty} dx\, f(x) = \text{p.v.} \int_{-\infty}^{\infty} dx\, f(x)$.
But consider $f(x) = 1/x^3$.

$$\int_{-\infty}^{\infty} dx\, \frac{1}{x^3} = \int_{-\infty}^{0} dx\, \frac{1}{x^3} + \int_{0}^{\infty} dx\, \frac{1}{x^3} = \infty + (-\infty) \quad \text{not well-defined}$$

$$\text{p.v.} \int_{-\infty}^{\infty} dx\, \frac{1}{x^3} = \lim_{\substack{R\to\infty \\ \varepsilon\to 0}} \int_{-R}^{-\varepsilon} dx\, \frac{1}{x^3} + \int_{\varepsilon}^{R} dx\, \frac{1}{x^3} = \lim_{\substack{R\to\infty \\ \varepsilon\to 0}} \frac{-1}{2}\underbrace{\left[+\frac{1}{\varepsilon^2} - \frac{1}{R^2} + \frac{1}{R^2} - \frac{1}{\varepsilon^2}\right]}_{=0}$$

well-defined.

example: $\text{p.v.} \int_{-\infty}^{\infty} dx\, \frac{e^{ikx}}{x-a}$ where $a>0$ is a real number, $k>0$ real



$$\text{p.v.} \int_{-\infty}^{\infty} dx\, \frac{e^{ikx}}{x-a} = \int_{P_1} dz\, \frac{e^{ikz}}{z-a} + \int_{P_2} dz\, \frac{e^{ikz}}{z-a}$$

Consider the contour $C = P_1 + P_2 + P_3 + P_4$.

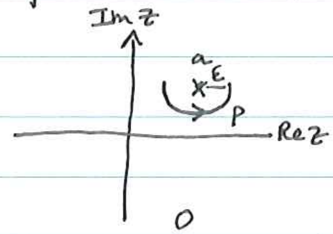
$$\oint_C dz\, \frac{e^{ikz}}{z-a} = 2\pi i \,\text{Res}\,(f(a)) = 2\pi i\, e^{ika}$$

$$\int_{P_4} dz\, \frac{e^{ikz}}{z-a} = 0$$

Need to compute: $\int_{P_3} dz\, \frac{e^{ikz}}{z-a}$.

Similar to our original calculation of the residue, but it is a half-circle, not a full (closed) circle.

Given a function $f(z) = \frac{g(z)}{z-a}$ with a simple pole at $z=a$ and path P located at $|z-a| = \varepsilon$, going from $\theta = -\pi$ to $\theta = 0$:



$$z = a + \varepsilon e^{i\theta}$$
$$dz = \varepsilon e^{i\theta}\, i\, d\theta$$

$$\lim_{\varepsilon\to 0} \int_P dz\, f(z) = \lim_{\varepsilon} \int_{-\pi}^{0} d\theta\, i\varepsilon e^{i\theta} \frac{g(a+\varepsilon e^{i\theta})}{\varepsilon e^{i\theta}} = g(a)\, i\pi = \pi i \operatorname{Res} f(a)$$

So a (counter-clockwise) half-circle picks up $\pi i \times$ Residue, i.e. half the contribution of a closed circle.

$$\text{So} \quad \int_{P_3} dz\, \frac{e^{ikz}}{z-a} = \pi i \operatorname{Res} f(a)$$

Putting everything together:

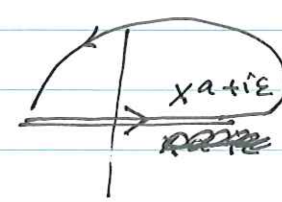
$$\text{P.V.} \int_{-\infty}^{\infty} dx\, \frac{e^{ikx}}{x-a} = \oint_C dz\, \frac{e^{ikz}}{z-a} - \int_{P_3} dz\, \frac{e^{ikz}}{z-a} - \int_{P_4} dz\, \frac{e^{ikz}}{z-a}$$

$$= 2\pi i \operatorname{Res} f(a) - \pi i \operatorname{Res} f(a) - 0$$

$$= \pi i\, e^{ika}$$

In physics applications, it is not always the P.V. which is the desired prescription for an integral. The pole at $x=a$ can be "regulated" by modifying the integrand:

$$\frac{e^{ikz}}{z-a} \to \frac{e^{ikz}}{z-a-i\varepsilon}$$



and then taking the limit $\varepsilon \to 0$.

$$\lim_{\varepsilon\to 0} \int_{-\infty}^{\infty} dx\, \frac{e^{ikx}}{x-a-i\varepsilon} = \lim_{\varepsilon\to 0} \oint dz\, \frac{e^{ikz}}{z-a-i\varepsilon} = 2\pi i \operatorname{Res} f(a+i\varepsilon)$$

$$= 2\pi i\, e^{ika}$$

Review: complex analysis

Cauchy's theorem: $\oint_C dz\, f(z) = 0$ if $f(z)$ is analytic on and within C

Contour deformation: given any two paths P_1 and P_2 starting at z_1 and ending at z_2, $\int_{P_1} dz\, f(z) = \int_{P_2} dz\, f(z)$ if $f(z)$ is analytic on and within the combined path $P_1 - P_2$.
(The contour P_1 can be deformed into P_2 without changing the integral.)

Residue theorem: $\oint_C dz\, f(z) = 2\pi i \sum_{i=1}^{N} \mathrm{Res}\, f(a_i)$ where $f(z)$ is analytic on and within C except for N isolated singularities at $z = a_1, \ldots, a_N$.

Computing the residue: suppose $f(z)$ has isolated singularity at $z = a$ of order n.

(1) Cauchy formula: $\mathrm{Res}\, f(a) = \lim_{z \to a} \frac{1}{(n-1)!} \frac{d^{n-1}}{dz^{n-1}} \left((z-a)^n f(z) \right)$
- poles

(2) Laurent expansion: $\mathrm{Res}\, f(a) = c_{-1}$ where c_{-1} is the coeffient of $\frac{1}{z-a}$ in Laurent expansion of $f(z)$ about $z = a$.
- essential singularities

Trig integrals $\int_0^{2\pi} d\theta$: express as integral in complex plane around unit circle ($|z| = 1$)

Real integral $\int_{-\infty}^{\infty} dx$:
- close the contour with a semicircle in the + or − imaginary half plane; check that the integral vanish along semicircle
- poles along the real axis must be avoided with a semicircle indent (principal value) or another prescription.