V. Fourier Transform

For a function $f(x)$, the Fourier transform is

$$\mathcal{F}[f(x)] = F(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx \; e^{-ikx} f(x)$$

Like the Laplace transform, the Fourier transform is an integral transform.

Fourier transform is similar to Fourier series. A function $f(x)$ defined over $-L \leq x \leq L$ can be expanded as

$$f(x) = \sum_{n=-\infty}^{\infty} c_n \, e^{in\pi x/L}$$

where the Fourier series coefficient is

$$c_n = \frac{1}{2L} \int_{-L}^{L} dx \; f(x) \, e^{-in\pi x/L}$$

Fourier transform generalizes Fourier series to functions $f(x)$ defined over $-\infty < x < \infty$. That is, taking $L \to \infty$.

For Fourier series, the wave number is $k_n = \frac{n\pi}{L}$. The spacing between wavenumbers is $\Delta k = k_{n+1} - k_n = \frac{\pi}{L}$.

For Fourier transform, we want to consider $L \to \infty$, so $\Delta k \to 0$. Therefore, we need to label Fourier modes by a continuous variable k (wavenumber), rather than a discrete index n.

Inverse Fourier transform:

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk \; e^{ikx} \; F(k)$$

This is similar to the Fourier series

$$f(x) = \sum_{n=-\infty}^{\infty} C_n \, e^{in\pi x/L}$$

where $\sum_n \rightarrow \int dk$

$C_n \rightarrow F(k)$

$e^{i\pi n x/L} \rightarrow e^{ikx}$

Let's prove that the inverse Fourier transform is correct.
First, we need to prove a useful result:

$$\delta(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk \; e^{ikx}$$

Proof: use Laplace transform $\mathcal{L}[\delta(x)] = \int_0^{\infty} dx \, e^{-sx} \, \delta(x) = 1$

Can invert the Laplace transform using Bromwich integral:

$$\delta(x) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} ds \; e^{sx} \cdot 1$$

Next, let $s = ik$. Then $ds = i\,dk$

$$\delta(x) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} dk \; i \, e^{ikx} = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk \; e^{ikx}$$

More general result:

$$\delta(x - x') = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk \; e^{ik(x-x')}$$

Now we can prove the inverse Laplace transform:

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk\, e^{ikx} F(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk\, e^{ikx} \int_{-\infty}^{\infty} dx'\, e^{-ikx'} f(x')$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} dx' \int_{-\infty}^{\infty} dk\, e^{ik(x-x')} f(x') = \frac{1}{2\pi} \int_{-\infty}^{\infty} dx' (2\pi)\, \delta(x-x') f(x')$$

$$= f(x)$$

Or we can show the same result for $F(k)$:

$$F(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx\, e^{-ikx} f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dx\, e^{-ikx} \int_{-\infty}^{\infty} dk'\, e^{ik'x} F(k')$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} dk' \underbrace{\int_{-\infty}^{\infty} dx\, e^{i(k'-k)x}}_{=(2\pi)\,\delta(k-k')} F(k')$$

$$= \int_{-\infty}^{\infty} dk'\, \delta(k-k') F(k') = F(k)$$

Note: the factors of $\frac{1}{\sqrt{2\pi}}$ are a convention. Alternate convention:

$$F(k) = \int_{-\infty}^{\infty} dx\, e^{-ikx} f(x)$$

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk\, e^{ikx} F(k)$$

The product of the prefactors must be $\frac{1}{2\pi}$.

Like the Laplace transform, the Fourier transform is a linear operator:

$$(1)\quad \mathcal{F}[c f(x)] = c\,\mathcal{F}[f(x)]$$

$$(2)\quad \mathcal{F}[f_1(x) + f_2(x)] = \mathcal{F}[f_1(x)] + \mathcal{F}[f_2(x)]$$

Also, Fourier transform must converge to be well-defined.
$f(x)$ shouldn't blow-up for $x \to \pm\infty$.

example: $f(x) = 1$

$$F(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx\, e^{-ikx} = \frac{1}{\sqrt{2\pi}} (2\pi)\, \delta(k) = \sqrt{2\pi}\, \delta(k)$$

example: $f(x) = e^{iax}$ where a is real

$$F(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx\, e^{-ikx}\, e^{iax} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx\, e^{i(a-k)x}$$

$$= \sqrt{2\pi}\, \delta(a-k) = \sqrt{2\pi}\, \delta(k-a)$$

example: $f(x) = e^{-\alpha|x|}$ where α is real and positive

$$F(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx\, e^{-ikx}\, e^{-\alpha|x|}$$

$$= \frac{1}{\sqrt{2\pi}} \left[\int_{-\infty}^{0} dx\, e^{(-ik+\alpha)x} + \int_{0}^{\infty} dx\, e^{(-ik-\alpha)x} \right]$$

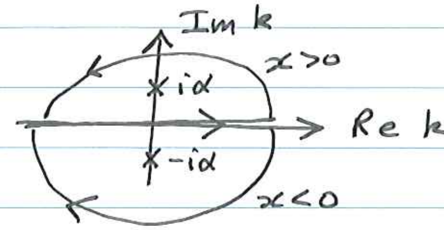
$$= \frac{1}{\sqrt{2\pi}} \left[\frac{1}{-ik+\alpha} - \frac{1}{-ik-\alpha} \right]$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{\alpha+ik + \alpha-ik}{\alpha^2+k^2} \right] = \sqrt{\frac{2}{\pi}}\, \frac{\alpha}{\alpha^2+k^2}$$

inverse Fourier transform:

$$f(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}dk\; e^{ikx}F(k)=\frac{1}{\pi}\int_{-\infty}^{\infty}dk\;\frac{e^{ikx}\alpha}{\alpha^2+k^2}=\frac{1}{\pi}\int_{-\infty}^{\infty}dk\;\frac{e^{ikx}\alpha}{(k-i\alpha)(k+i\alpha)}$$

use contour integration:



poles at $k=\pm i\alpha$

close ~~pole~~ contour above for $x>0$.
close contour below for $x<0$.

case i: $x>0$

$$f(x)=\frac{1}{\pi}\,2\pi i\,\mathrm{Res}(i\alpha)$$

$$=2i\lim_{k\to i\alpha}\cancel{(k-i\alpha)}\frac{\alpha e^{ikx}}{\cancel{(k-i\alpha)}(k+i\alpha)}=2i\,\frac{\alpha e^{-\alpha x}}{2i\alpha}=e^{-\alpha x}$$

$$=e^{-\alpha x}$$

case ii: $x<0$ $\rightarrow$ similar but $f(x)=\frac{1}{\pi}(-1)2\pi i\,\mathrm{Res}(-i\alpha)$
$=e^{\alpha x}$

So $f(x)=e^{-\alpha|x|}$

In quantum mechanics, the momentump is related to wavenumber k by $p = \hbar k$.

Fourier transform is useful for transforming between the position basis and the momentum basis wavefunctions.

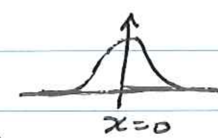
$\Psi(x) =$ position basis wavefunction

$\mathcal{F}[\Psi(x)] = \tilde{\Psi}(k) =$ momentum basis wavefunction.

↑alternate notation for Fourier transform.

Example: Gaussian wave packet.

Consider a wavefunction $\Psi(x) = N e^{-\alpha^2 x^2/2}$



This represents a "free" particle (potential $V=0$) localized at $x=0$.

Normalization: fix N using $\int_{-\infty}^{\infty} dx\, |\Psi(x)|^2 = 1$

$$\Rightarrow \int_{-\infty}^{\infty} dx\, N^2 e^{-\alpha^2 x^2} = N^2 \frac{\sqrt{\pi}}{\alpha} = 1 \quad \text{so} \quad N = \sqrt{\frac{\alpha}{\sqrt{\pi}}}$$

Gaussian integral:

$$\left(\int_{-\infty}^{\infty} dx\, e^{-\alpha^2 x^2}\right)^2 = \int_{-\infty}^{\infty} dx\, e^{-\alpha^2 x^2} \int_{-\infty}^{\infty} dy\, e^{-\alpha^2 y^2} = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy\, e^{-\alpha^2 (x^2+y^2)}$$

now write in polar coordinates r, θ.

$$\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \rightarrow \int_0^{\infty} r\, dr \int_0^{2\pi} d\theta$$

$$x^2 + y^2 \rightarrow r^2$$

$$= \int_0^{\infty} r\,dr \int_0^{2\pi} d\theta\, e^{-\alpha^2 r^2} = 2\pi \int_0^{\infty} r\,dr\, e^{-\alpha^2 r^2} = \frac{\pi}{-\alpha^2}\, e^{-\alpha^2 r^2}\Big|_0^{\infty} = \frac{\pi}{\alpha^2}$$

$$\Rightarrow \int_{-\infty}^{\infty} dx\, e^{-\alpha^2 x^2} = \frac{\sqrt{\pi}}{\alpha}$$

Another useful Gaussian integral:

$$\int_{-\infty}^{\infty} dx\, x^2 e^{-\alpha^2 x^2} = \frac{\sqrt{\pi}}{2\alpha^3}$$

Proof: note: $\frac{\partial}{\partial\alpha}\left(\int_{-\infty}^{\infty} dx\, e^{-\alpha^2 x^2}\right) = -2\alpha \int_{-\infty}^{\infty} dx\, e^{-\alpha^2 x^2} x^2 = -\frac{\sqrt{\pi}}{\alpha^2}$

The width of the wave packet represents the uncertainty in x.

$$\Delta x = \sqrt{\langle x^2\rangle - \langle x\rangle^2} \qquad \text{uncertainty in } x$$

$$\langle x^2\rangle = \int_{-\infty}^{\infty} dx\, x^2 |\psi(x)|^2 = \int_{-\infty}^{\infty} dx\, N^2 x^2 e^{-\alpha^2 x^2}$$

$$= N^2 \frac{\sqrt{\pi}}{2\alpha^3} = \frac{1}{2\alpha^2}$$

$\langle x\rangle = 0$ since integrand is odd under $x \to -x$.

So $\Delta x = \frac{1}{\sqrt{2}\alpha}$. As $\alpha \to \infty$, particle move localized at $x=0$

As $\alpha \to 0$, less localized

Next, consider the Fourier transform:

$$\tilde{\Psi}(k) = \int_{-\infty}^{\infty} dx\, e^{-ikx}\, \Psi(x) = \int_{-\infty}^{\infty} dx\, e^{-ikx} N e^{-\alpha^2 x^2/2}$$

$$= N \int_{-\infty}^{\infty} dx\, e^{-\frac{\alpha^2 x^2}{2} - ikx}$$

Trick: complete the square

$$\frac{\alpha^2 x^2}{2} + ikx = \frac{\alpha^2}{2}\left(x^2 + \frac{2ik}{\alpha^2}x + \left(\frac{ik}{\alpha^2}\right)^2\right) - \frac{\alpha^2}{2}\left(\frac{ik}{\alpha^2}\right)^2$$

$$= \frac{\alpha^2}{2}\left(x + \frac{ik}{\alpha^2}\right)^2 + \frac{k^2}{2\alpha^2}$$

$$\tilde{\Psi}(k) = \frac{N}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx \; e^{-\frac{k^2}{2\alpha^2}} \; e^{-\frac{\alpha^2}{2}(x + ik/\alpha^2)^2}$$

Let $u = x + \frac{ik}{\alpha^2}$

$du = dx$

Then $\tilde{\Psi}(k) = \frac{N}{\sqrt{2\pi}} e^{-\frac{k^2}{2\alpha^2}} \int_{-\infty + \frac{ik}{\alpha^2}}^{\infty + ik/\alpha^2} du \; e^{-\frac{\alpha^2}{2}u^2} = \frac{N}{\sqrt{2\pi}} e^{-\frac{k^2}{2\alpha^2}} \frac{\sqrt{2\pi}}{\alpha}$

$$= \frac{N}{\alpha} e^{-\frac{k^2}{2\alpha^2}}$$

is the momentum basis wave function.

The uncertainty in the momentum is:

$$\Delta p = \hbar \Delta k = \hbar \sqrt{\langle k^2 \rangle - \langle k \rangle^2}$$

$$\langle k^2 \rangle = \int_{-\infty}^{\infty} dk \, |\tilde{\Psi}(k)|^2 = \int_{-\infty}^{\infty} dk \, \frac{N^2}{\alpha^2} e^{-\frac{k^2}{\alpha^2}} k^2 = \frac{N^2}{\alpha^2} \frac{\sqrt{\pi}\,\alpha^3}{2}$$

$$= \frac{\alpha}{\sqrt{\pi}} \frac{\sqrt{\pi}\,\alpha}{2} = \frac{\alpha^2}{2}$$

$\langle k \rangle = 0$ (integrand odd under $k \to -k$)

$$\Delta p = \hbar \Delta k = \hbar \frac{\alpha}{\sqrt{2}}$$

Uncertainty principle: $\Delta x = \frac{1}{\sqrt{2}\alpha}$, $\Delta p = \hbar \frac{\alpha}{\sqrt{2}}$

$\alpha \to 0$: $\Delta x \to \infty$, but $\Delta p \to 0$ (well-defined in momentum)

$\alpha \to \infty$: $\Delta x \to 0$, but $\Delta p \to \infty$ (well-defined in position)

$$\Delta x \, \Delta p = \frac{\hbar}{2}$$

Gaussian wavepacket saturates the uncertainty principle $\Delta x \Delta p \geq \hbar/2$

Fourier transform to solve differential equations

Fourier transform of derivatives:

Function $f(x)$ has Fourier transform $F(k) = \mathcal{F}[f(x)]$

Then $\mathcal{F}\left[\frac{\partial f}{\partial x}\right] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx\, e^{-ikx} \frac{\partial f}{\partial x}$ integrate by parts

$$= \frac{1}{\sqrt{2\pi}} \left(\underbrace{e^{-ikx} f(x) \Big|_{-\infty}^{\infty}}_{\to\, 0 \text{ if } f(\pm\infty) \to 0} + ik \int_{-\infty}^{\infty} dx\, e^{-ikx} f(x) \right)$$

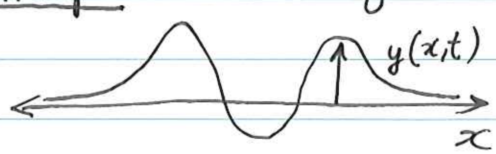
$$= ik\, F(k)$$

So $\mathcal{F}\left[\frac{\partial f}{\partial x}\right] = (ik)\, \mathcal{F}[f(x)] = ik\, F(k)$

More general: $\mathcal{F}\left[\frac{\partial^n f}{\partial x^n}\right] = (ik)^n \mathcal{F}[f(x)] = (ik)^n F(k)$

Derivatives $\frac{\partial}{\partial x} \to ik$

Example: wave equation for an infinite string



$$\frac{\partial^2 y}{\partial t^2} = v^2 \frac{\partial^2 y}{\partial x^2}$$

Fourier transform with respect to x:

$$Y(k,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx\, e^{-ikx}\, y(x,t)$$

Y is still a function of time.

Fourier transform the wave equation:

$$\frac{\partial^2}{\partial t^2} Y(k,t) = v^2 (ik)^2 Y(k,t) = -v^2 k^2 Y(k,t)$$

$Y(k,t)$ satisfies equation for simple harmonic oscillator with frequency $\omega = vk$.

So we have: $\cancel{Y(k,t) = A_k e^{i\omega t} + B_k}$

$$Y(k,t) = A(k)\, e^{-i\omega t} + B(k)\, e^{+i\omega t}$$

The coefficients A & B are functions of k and depend on the initial condition $y(x,0)$, $\dot{y}(x,0)$.

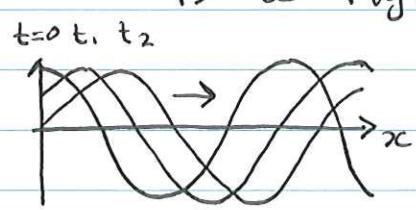
The final solution for $y(x,t)$ is obtained by the inverse Fourier transform:

$$y(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk\, e^{ikx}\, Y(k,t)$$

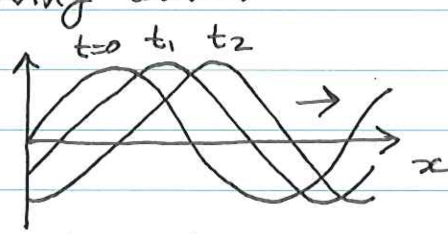
$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk \left(A(k)\, e^{-i(\omega t - kx)} + B(k)\, e^{i(\omega t + kx)} \right)$$

Note: $e^{-i(\omega t - kx)} = \cos(\omega t - kx) - i \sin(\omega t - kx)$
$= \cos(kx - \omega t) + i \sin(kx - \omega t)$

is a right-moving wave.



$\cos(kx - \omega t)$



$\sin(kx - \omega t)$

In time Δt, the crest of the wave moves a distance

$$\Delta x = \frac{\omega}{k} \Delta t = \frac{vk}{k} \Delta t = v \Delta t$$

The <u>wave velocity</u> $\frac{\Delta x}{\Delta t} = v$ is the velocity of this individual Fourier mode k.

Likewise, $e^{+i(\omega t + kx)}$ is a left-moving wave.
The distance traveled is

$$\Delta x = -\frac{\omega}{k} \Delta t = -v \Delta t$$

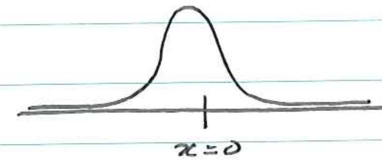
So the wave velocity is $-v$.

All Fourier modes travel with same wave velocity (up to a sign)

General formula: $\text{velocity} = \frac{\partial x}{\partial t} = \frac{\partial x / \partial y}{\partial t / \partial y} = \frac{\frac{\partial y}{\partial t}}{\frac{\partial y}{\partial x}} = \frac{\omega}{k} = \pm v$

for a given Fourier mode.

Next, consider an initial condition:



$$y(x,0) = e^{-\alpha^2 x^2/2}, \quad \dot{y}(x,0) = 0$$

What is $y(x,t)$?

Compute $$Y(k,0) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}}\, dx\; e^{-\alpha^2 x^2/2}\, e^{-ikx}$$

$$= \frac{1}{\alpha} e^{-\frac{k^2}{2\alpha^2}}$$

$$\dot{Y}(k,0) = \mathcal{F}[0] = 0$$

$$Y(k,t) = A(k)\, e^{-i\omega t} + B(k)\, e^{+i\omega t}$$

$$Y(k,0) = A(k) + B(k) = \frac{1}{\alpha} e^{-\frac{k^2}{2\alpha^2}}$$

$$\dot{Y}(k,0) = i\omega\,(B(k) - A(k)) = 0$$

So $$A(k) = B(k) = \frac{1}{2\alpha} e^{-\frac{k^2}{2\alpha^2}}$$

$$y(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk\, \frac{1}{2\alpha}\, e^{-k^2/2\alpha^2} \left(e^{-i(\omega t - kx)} + e^{i(\omega t + kx)} \right)$$

$$= \frac{1}{\sqrt{2\pi}\, 2\alpha} \int_{-\infty}^{\infty} dk \left(e^{-\frac{k^2}{2\alpha^2} - ik(vt - x)} + e^{-\frac{k^2}{2m} + i(vt+x)k} \right)$$

Complete the square: right moving wave term

$$-\frac{k^2}{2\alpha^2} - ik(vt-x) = -\frac{1}{2\alpha^2}\left(k^2 + 2\alpha^2 i k (vt - x)\right)$$

$$= -\frac{1}{2\alpha^2}\left[(k + i\alpha^2(vt-x))^2 - (i\alpha^2(vt-x))^2 \right]$$

$$= -\frac{1}{2\alpha^2}\left[u^2 + \alpha^4 (vt-x)^2\right]$$

$$= -\frac{u^2}{2\alpha^2} - \frac{\alpha^2 (vt-x)^2}{2}$$

where $u = k + i\alpha^2 (vt-x)$

right-moving term:

$$y(x,t) = \frac{1}{\sqrt{2\pi}\, 2\alpha} \int_{-\infty + i\alpha^2 (vt-x)}^{\infty + i\alpha^2(vt-x)} du \; e^{-\frac{u^2}{2\alpha^2}} \, e^{-\frac{\alpha^2 (vt-x)^2}{2}}$$

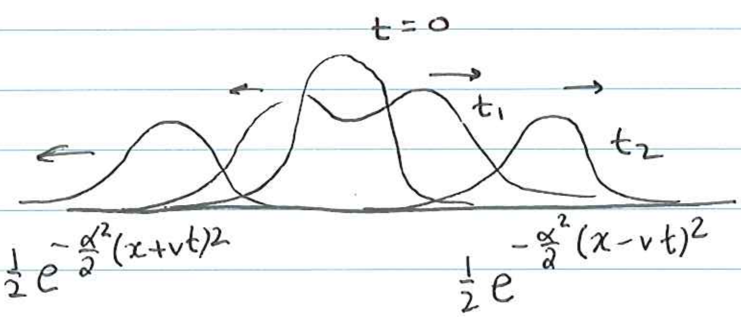
$$= \frac{1}{\sqrt{2\pi}\, 2\alpha}\left(\sqrt{2\pi}\,\alpha\right) e^{-\frac{\alpha^2}{2}(vt-x)^2}$$

left-moving term: same, but $v \to -v$

total expression:

$$y(x,t) = \frac{1}{2}\left(e^{-\frac{\alpha^2}{2}(x-vt)^2} + e^{-\frac{\alpha^2}{2}(x+vt)^2}\right)$$

Initial Gaussian splits into two traveling waves that keep the same shape of the initial displacement



Fourier transform with respect to time:

Consider a function $f(t)$ where t is time. The Fourier transform is:

$$\mathcal{F}[f(t)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dt\, e^{i\omega t} f(t) = F(\omega)$$

Inverse Fourier transform:

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\omega\, e^{-i\omega t} F(\omega)$$

By convention, the sign of the exponential is reversed compared to FT with respect to position x.

Fourier trans.

position x $\longleftrightarrow$ wavenumber k
momentum $p = \hbar k$

time t $\longleftrightarrow$ frequency ω
Energy $E = \hbar\omega$

example: infinite string $\frac{\partial^2 y}{\partial t^2} = -v^2 \frac{\partial^2 y}{\partial x^2}$

Define $Y(\omega, k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dt\, e^{i\omega t} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx\, e^{-ikx}\, y(x,t)$

double Fourier transform.

Note: $\mathcal{F}\left[\frac{\partial^n y}{\partial x^n}\right] = (ik)^n \mathcal{F}[y(x)]$

$\mathcal{F}\left[\frac{\partial^n y}{\partial t^n}\right] = (-i\omega)^n \mathcal{F}[y(t)]$

So the Fourier transform of the wave equation becomes:

$$(-i\omega)^2\, Y(\omega,k) = -v^2\,(ik)^2\, Y(\omega,k)$$

$$\Rightarrow \quad (\omega^2 - v^2k^2)\, Y(\omega,k) = 0$$

Thus either $\omega^2 = (vk)^2$ or $Y(\omega,k) = 0$.

- The relation $\omega^2 = (vk)^2$ is called a dispersion relation. It says what is the allowed frequency ω (or energy) for a given wavenumber k (or momentum).

Here we have $\omega = \pm vk$.

We must have: $Y(\omega,k) = A(k)\,\delta(\omega - vk) + B(k)\,\delta(\omega + vk)$

$$\text{Then } y(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\, e^{-i\omega t}\int_{-\infty}^{\infty} dk\, e^{ikx}\, Y(\omega,k)$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} dk \left(e^{-i(vkt-kx)}\, A(k) + e^{i(vkt+kx)}\, B(k)\right)$$

Same answer as before (upto factor of $1/\sqrt{2\pi}$)

The struck (infinite) string

Wave equation $\quad \frac{\partial^2 y}{\partial t^2} = v^2 \frac{\partial^2 y}{\partial x^2}$

Solution: $\quad y(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk \left(A(k) e^{-i\omega t + ikx} + B(k) e^{i\omega t + ikx} \right)$

$$\omega = vk$$

Now consider "striking" the string at $x=0$ at $t=0$:

$$y(x,0) = 0 \qquad \dot{y}(x,0) = \delta(x)$$

impulse at $x=0$

$$Y(k,0) = A(k) + B(k) = 0 \quad \rightarrow \quad A(k) = -B(k)$$

$$\dot{Y}(k,0) = -i\omega \left(A(k) - B(k) \right) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk\, e^{-ikx} \delta(x) = \frac{1}{\sqrt{2\pi}}$$

So $\quad A(k) = -B(k) = \frac{-1}{2i\omega} \frac{1}{\sqrt{2\pi}}$

$$y(x,t) = \frac{1}{2\pi} \frac{-1}{2i} \int_{-\infty}^{\infty} dk \frac{1}{vk} \left(e^{-i\omega t + ikx} - e^{i\omega t + ikx} \right)$$

$$= \frac{1}{2\pi i} \int_{-\infty}^{\infty} dk \frac{1}{2vk} \left(e^{i(vt+x)k} - e^{i(x-vt)k} \right)$$

Do $\int_{-\infty}^{\infty} dk$ integral by contour integration.

Pole at $k=0$. Note: this looks like a removable singularity since

$$\lim_{k \to 0} \frac{1}{k} \left(e^{i(x+vt)k} - e^{i(x-vt)k} \right) = \text{finite}$$

(doesn't blow up)

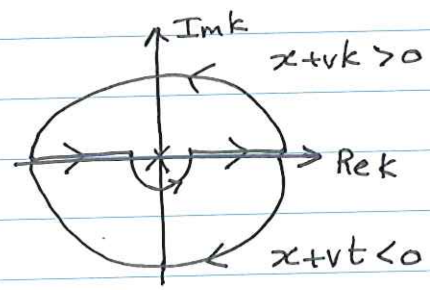
But the situation is more subtle since we need to ensure exponentials vanish when we close the contour.

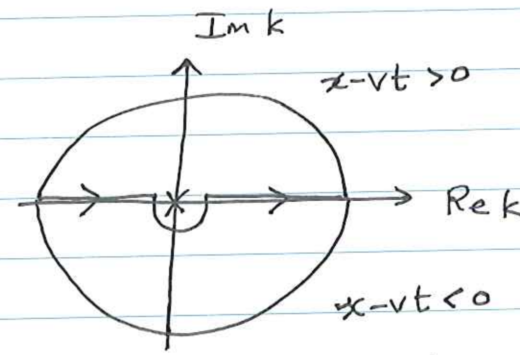
Two terms:

$$\frac{1}{k} e^{i(x+vt)k} \quad \text{and} \quad \frac{1}{k} e^{i(x-vt)k}$$

$$\mathrm{Res}(k=0) = 1 \qquad\qquad \mathrm{Res}(k=0) = 1$$

When both contours close the same way, the residues cancel out.

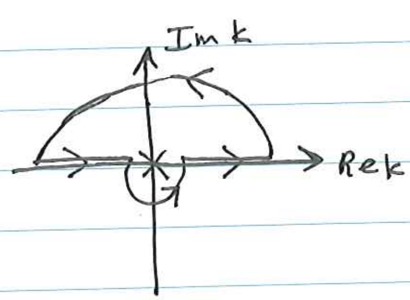




$$e^{i(x+vt)k} \qquad\qquad e^{i(x-vt)k}$$

Only non-zero case: $vt > x > -vt$ or $|x| < vt$.

Then $x+vt > 0$ but $x-vt < 0$.



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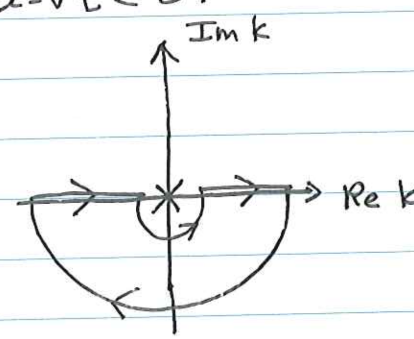


$$2\pi i\, \mathrm{Res} - \pi i\, \mathrm{Res} \quad - \quad -i\pi\, \mathrm{Res}$$

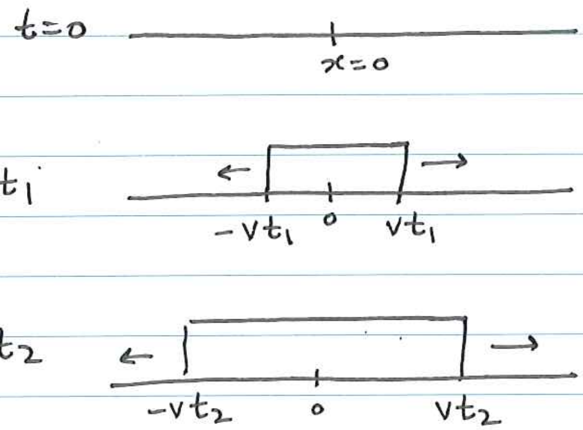
$$= 2\pi i\, \mathrm{Res}(k=0) = 2\pi i$$

So we have

$$y(x,t) = \frac{1}{2\pi i}\,\frac{1}{2v}\cdot 2\pi i = \frac{1}{2v}$$

but only for $vt > |x|$. So the solution is:

$$y(x,t) = \frac{1}{2v}\,\theta(vt - |x|)$$



Green's functions

Fourier transform can be useful to compute Green's functions.

example: forced harmonic oscillator

$$\ddot{x}(t) + \omega_0^2 x(t) = f(t) \quad \text{forcing term}$$

Recall: Green's function $g(t,\tau)$ satisfies

$$\ddot{g}(t,\tau) + \omega_0^2 g(t,\tau) = \delta(t-\tau)$$

Solve for $g(t,\tau)$, then the solution for $x(t)$ is:

$$x(t) = \int_{-\infty}^{\infty} d\tau \; g(t,\tau) f(\tau)$$

$$\left(\frac{d^2}{dt^2} + \omega_0^2\right) x(t) = \left(\frac{d^2}{dt^2} + \omega_0^2\right) \int_{-\infty}^{\infty} d\tau \; g(t,\tau) f(\tau) = \int_{-\infty}^{\infty} d\tau \; \delta(t-\tau) f(\tau) = f(t)$$

Compute $g(t,\tau)$ using Fourier transform:

$$G(\omega,\tau) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dt \; e^{i\omega t} g(t,\tau)$$

Take F.T. of diff. eqn. for g:

$$(-i\omega)^2 G(\omega,\tau) + \omega_0^2 G(\omega,\tau) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dt \; e^{i\omega t} \delta(t-\tau) = \frac{1}{\sqrt{2\pi}} e^{i\omega\tau}$$

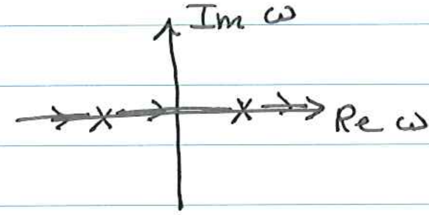
$$G(\omega,\tau) = -\frac{1}{\sqrt{2\pi}} \frac{e^{i\omega\tau}}{\omega^2 - \omega_0^2}$$

Then take inverse Fourier transform:

$$g(t,\tau) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\omega \, G(\omega,\tau) e^{-i\omega t} = -\frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega \, \frac{e^{-i\omega(t-\tau)}}{\omega^2 - \omega_0^2}$$

compute the integral using contour integration:

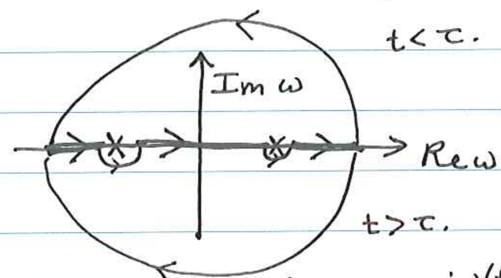
poles at $\omega = \pm\omega_0$



Poles along the real axis. Need a prescription for avoiding the poles.

Need to put in some physics to get the right prescription.

1. First idea: take principal value (go around the poles)



case: $t>\tau$: $\displaystyle\int_{-\infty}^{\infty} d\omega \, \frac{e^{-i\omega(t-\tau)}}{\omega^2-\omega_0^2} = 0 - \pi i \, Res(+\omega_0) - \pi i \, Res(-\omega_0)$

case: $t<\tau$: $\displaystyle\int_{-\infty}^{\infty} d\omega \, \frac{e^{-i\omega(t-\tau)}}{\omega^2-\omega_0^2} = 2\pi i \, Res(\omega_0) + 2\pi i \, Res(-\omega_0)$
$\quad - \pi i \, Res(\omega_0) - \pi i \, Res(-\omega_0)$
$\quad = \pi i \, Res(\omega_0) + \pi i \, Res(-\omega_0)$

$$Res(+\omega_0) = \frac{e^{-i\omega_0(t-\tau)}}{2\omega_0}$$

$$Res(-\omega_0) = -\frac{e^{i\omega_0(t-\tau)}}{2\omega_0}$$

$$\mathrm{Res}(\omega_0)+\mathrm{Res}(-\omega_0)=\frac{1}{2\omega_0}\left(e^{-i\omega_0(t-\tau)}-e^{i\omega_0(t-\tau)}\right)$$

$$=\frac{-2i}{2\omega_0}\sin(\omega_0(t-\tau))$$

$$g(t,\tau)=-\frac{1}{2\pi}\left(-\frac{2i}{2\omega_0}\right)\begin{cases}(-i\pi)\sin(\omega_0(t-\tau)) & t>\tau\\ (i\pi)\sin(\omega_0(t-\tau)) & t<\tau\end{cases}$$

$$=\begin{cases}\frac{1}{2\omega_0}\sin(\omega_0(t-\tau)) & t>\tau\\ -\frac{1}{2\omega_0}\sin(\omega_0(t-\tau)) & t<\tau\end{cases}$$

Green's function gets contributions from both after $(t>\tau)$ and before $(t<\tau)$ the impulse.

$$x(t)=\int_{-\infty}^{\infty}d\tau\, g(t,\tau)\, f(\tau)$$

$$=\int_{-\infty}^{t}d\tau\,\frac{1}{2\omega_0}\sin(\omega_0(t-\tau))f(\tau)-\int_{t}^{\infty}d\tau\,\frac{1}{2\omega_0}\sin(\omega_0(t-\tau))f(\tau)$$

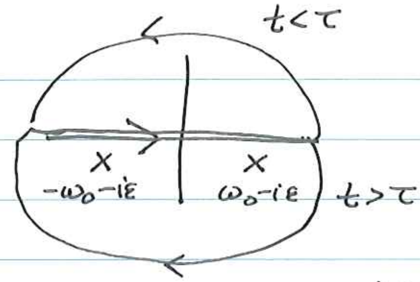
contribution to $x(t)$ from times $\tau>t$. Violates causality.

2. Alternate prescription: move the poles

$$\int_{-\infty}^{\infty}d\omega\,\frac{e^{-i\omega(t-\tau)}}{\omega^2-\omega_0^2}=\lim_{\varepsilon\to 0}\int_{-\infty}^{\infty}d\omega\,\frac{e^{-i\omega(t-\tau)}}{\omega^2-\omega_0^2+2i\varepsilon\omega}$$

poles at $\omega^2+2i\varepsilon\omega-\omega_0^2=0$

$$\omega=\frac{1}{2}\left(-2i\varepsilon\pm\sqrt{(4i\varepsilon)^2+4\omega_0^2}\right)=\pm\omega_0-i\varepsilon$$



case $t > \tau$: $\displaystyle\int_{-\infty}^{\infty} d\omega \, \frac{e^{-i\omega(t-\tau)}}{\omega^2 - \omega_0^2 + 2i\varepsilon\omega} = (-2\pi i)\left(\mathrm{Res}(\omega_0 - i\varepsilon) + \mathrm{Res}(-\omega_0 - i\varepsilon)\right)$

$$= -\frac{2\pi}{\omega_0} \sin(\omega_0(t-\tau))$$

case $t < \tau$: $\displaystyle\int_{-\infty}^{\infty} d\omega \, \frac{e^{-i\omega(t-\tau)}}{\omega^2 - \omega_0^2 + 2i\varepsilon\omega} = 0$

So $g(t,\tau) = \begin{cases} \frac{1}{\omega_0} \sin(\omega_0(t-\tau)) & t > \tau \\ 0 & t < \tau \end{cases}$

And $x(t) = \displaystyle\int_{-\infty}^{\infty} d\tau \, g(t,\tau) f(\tau) = \int_{-\infty}^{t} d\tau \, g(t,\tau) f(\tau)$

This Green's function obeys causality since $x(t)$ only depends on $f(\tau)$ at $\tau < t$.