PHYS 3090: Homework 4 (due Wednesday Oct. 24)

Problem 1: Suppose we have the sawtooth wave with period $L = 1$ as shown below. What are the Fourier series coefficients a_n, b_n, and c_n? **(5 points)**

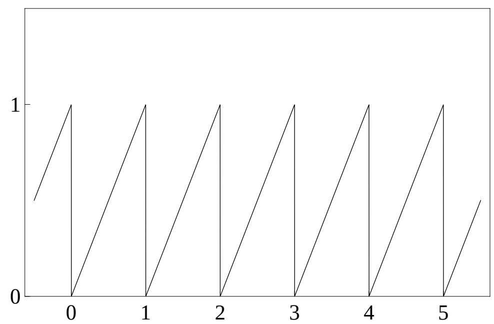

Figure 1: Sawtooth wave.

Problem 2: Given a function $f(x)$ with period L, **Parseval's theorem** states that

$$\frac{1}{L}\int_0^L |f(x)|^2 = \sum_{n=-\infty}^{\infty} |c_n|^2 = |b_0|^2 + \frac{1}{2}\sum_{n=1}^{\infty}\left(|a_n|^2 + |b_n|^2\right) \tag{1}$$

where a_n, b_n, c_n are the Fourier coefficients for $f(x)$.

(a) Prove Parseval's theorem. **(5 points)**

(b) Apply Parseval's theorem to the function in Problem 1 to derive the following sum **(5 points)**

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}\,. \tag{2}$$

Problem 3: Show that when a string of length L is plucked at a position L/k (where $k \geq 2$ is an integer), no Fourier modes of order $n = k, 2k, 3k, ...$ are excited. **(5 points)**

(This effect is utilized in pianos to eliminate the 7th harmonic by having the hammer strike the piano string at a position $L/7$.)

Problem 4: Let's consider the case of a **struck string**. A string of length L is fixed at its endpoints, at $x = 0$ and $x = L$, and is initially at rest for $t < 0$. Suppose at $t = 0$ that it is given an instantaneous impulse corresponding to the following initial condition:

$$y(x,0) = 0\,, \quad \dot{y}(x,0) = \begin{cases} 1 & \frac{L-d}{2} < x < \frac{L+d}{2} \\ 0 & \text{otherwise} \end{cases} \,. \tag{3}$$

This is interpreted as a hammer of length d striking the string centered at $x = L/2$ to impart an initial velocity at $t = 0$. Determine $y(x,t)$ for $t > 0$. **(5 points)**