PHYS 3090: Homework 5 (due Wed. Nov. 7)

Problem 1 (5 points): A function $f(t)$ has a Laplace transform

$$F(s) = \frac{s^2 - a^2}{(s^2 + a^2)^2} \,,$$

where a is a real, positive number. Compute $f(t)$ using the Bromwich integral.

Problem 2 (10 points):

(a) Let $f(t)$ be a function with Laplace transform $F(s)$. Prove the following: **(3 points)**

$$\int_s^\infty ds' \, F(s') = \mathcal{L}\big[f(t)/t\big] \,.$$

(b) Using the above result, compute $\mathcal{L}[\frac{\sin t}{t}]$. **(4 points)**

(c) Recall that the sine function is given by

$$\mathrm{Si}(a) = \int_0^a dt \, \frac{\sin t}{t} \,.$$

Using the results of parts (a) and (b), compute $\mathrm{Si}(\infty)$. **(3 points)**

Problem 3 (10 points): Suppose a radioactive isotope, with decay constant λ, starts leaking from a nuclear reactor at $t = 0$ with rate $R(t)$. The number of radioactive atoms present outside the leak satisfies the rate equation

$$\dot{n}(t) + \lambda \, n(t) = R(t) \,, \tag{1}$$

where λ is the decay rate. We also suppose the initial condition is $n(0) = 0$.

(a) Let's suppose that $R(t) = 0$ for $t < 0$ and that the leak "turns on" at $t = 0$. For $t \geq 0$, the leak rate is given by a periodic function $R(t) = R_0\big(1 - \cos(\omega t)\big)$, where R_0 and ω are positive, real constants. Determine $n(t)$ by taking the Laplace transform of Eq. (1). **(5 points)**

(b) Compute the Green's function $g(t, \tau)$ for the above rate equation and use this to write the solution $n(t)$ for arbitrary $R(t)$. For the rate $R(t)$ given in part (a), verify that the Green's function solution for $n(t)$ is the same as what you found in part (a). **(5 points)**

Problem 4 (5 points): In class, we showed that

$$\mathcal{L}[t^n] = \frac{n!}{s^{n+1}}$$

for any integer $n \geq 0$. In this problem, we will compute $\mathcal{L}[t^a]$ for a positive *non-integer* power a.

(a) Prove that

$$\mathcal{L}[t^a] = \frac{\Gamma(a+1)}{s^{a+1}} \, .$$

Here, we have introduced the following function

$$\Gamma(a) = \int_0^\infty dx \, x^{a-1} e^{-x} \, ,$$

which is known as the **gamma function** and cannot be expressed in terms of elementary functions. **(1 point)**

(b) Prove that $\Gamma(n+1) = n!$ for any integer $n \geq 0$. **(2 points)**

(c) The gamma function generalizes the notion of a factorial to numbers beyond integers. Recall that the factorial is defined recursively according to the relation $n! = n(n-1)!$ for integers $n > 0$. Show that the gamma function satisfies the same relation

$$\Gamma(a+1) = a\Gamma(a)$$

for any value of $a > 0$, i.e., not necessarily an integer. **(2 points)**

Problem 5 (10 points): The goal of this problem is to compute $\mathcal{L}[\ln^n t]$ for any integer power $n > 0$. Note $\ln^n t$ is a shorthand notation for $(\ln t)^n$.

(a) Show that

$$\mathcal{L}[\ln t] = -\frac{\gamma_E + \ln s}{s}$$

where γ_E is the Euler-Mascheroni constant, given by

$$\gamma_E = -\int_0^\infty dx \, e^{-x} \, \ln x = 0.5772156649\ldots \, .$$

Hint: This is simple to prove if you use the fact that $\ln t = \ln(st) - \ln s$ and make an appropriate change of variables. **(3 points)**

(b) Prove that the nth derivative of the gamma function, evaluated at $a = 1$, is given by

$$\Gamma^{(n)}(1) \equiv \lim_{a \to 1} \frac{\partial^n}{\partial a^n} \Gamma(a) = \int_0^\infty dx \, \ln^n x \, e^{-x} \, . \tag{2}$$

Note that $\Gamma'(1) = -\gamma_E$ **(2 points)**.

(c) Using Eq. (2), derive a general formula for $\mathcal{L}[\ln^n t]$. **(5 points)**