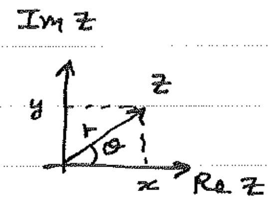
Complex analysis

Review of complex numbers



$$z = \underbrace{x + iy}_{\text{Cartesian form}} = \underbrace{r\,e^{i\theta}}_{\text{Polar form}} = r(\cos\theta + i\sin\theta)$$

real part $\quad x = \mathrm{Re}(z) = r\cos\theta$

imaginary part $\quad y = \mathrm{Im}(z) = r\sin\theta$

magnitude $\quad r = |z| = \sqrt{x^2+y^2}$

argument $\quad \theta = \arg(z) = \arctan(y/x)$

complex conjugate: $z^* = x - iy = r\,e^{-i\theta}$

Then $\quad |z|^2 = z^* z = z z^* = r^2 = x^2 + y^2$

Usual rules for arithmetic apply to complex numbers

$z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$

addition: $z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2)$

subtraction: $z_1 - z_2 = (x_1 - x_2) + i(y_1 - y_2)$

multiplication: $z_1 z_2 = (x_1 + iy_1)(x_2 + iy_2) = (x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1)$
$= r_1 r_2 e^{i(\theta_1+\theta_2)}$

division: $z_1/z_2 = \dfrac{z_1}{z_2}\dfrac{z_2^*}{z_2^*} = \dfrac{(x_1+iy_1)(x_2-iy_2)}{x_2^2+y_2^2}$

$$= \left(\frac{x_1x_2 + y_1y_2}{x_2^2+y_2^2}\right) + i\left(\frac{x_2y_1 - x_1y_2}{x_2^2+y_2^2}\right)$$

$$= \frac{r_1}{r_2}\,e^{i(\theta_1-\theta_2)}$$

Euler's formula: $e^{i\theta} = \cos\theta + i\sin\theta$

Can express trig functions as $\quad \cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta})$

$\sin\theta = \frac{1}{2i}(e^{i\theta} - e^{-i\theta})$

Similar to hyperbolic trig functions: $\quad \cosh x = \frac{1}{2}(e^{x} + e^{-x})$

$\sinh x = \frac{1}{2}(e^{x} - e^{-x})$

Function of complex variables

A complex function f takes a complex number $z = x + iy$, and returns another complex number $f(z)$

Can write $f(z) = \underbrace{u(x,y)}_{Re(f(z))} + i \underbrace{v(x,y)}_{Im(f(z))}$

Example: $f(z) = e^{z} = e^{x+iy} = e^{x}(\cos y + i \sin y)$

So $u = e^{x}\cos y$, $v = e^{x} \sin y$

example: $f(z) = \ln z = \ln(re^{i\theta}) = \ln r + \ln e^{i\theta}$

$$= \underset{u = \ln r}{\ln r} + i\underset{v=\theta}{\theta} = \ln|z| + i\arg(z)$$

Note: z is the same for $\theta \to \theta + 2\pi n$ $(n = 0, \pm 1, \pm 2, \dots)$ since $e^{2\pi i n} = 1$. But this changes

$$\ln z = \ln r + i\theta \to \ln r + i\theta + 2\pi i n$$

So $f(z) = \ln z$ is a multivalued function
It has infinitely many values.

Restrict $\ln z$ by specifying allowed range for θ (branch).

Principle branch: any branch that includes $\theta = 0$.

e.g. $z = -2i \Rightarrow r = |z| = 2$, $\theta = -\frac{\pi}{2} + 2\pi n$

Both principle branches.
$$\begin{cases} 0 \le \theta < 2\pi : & \theta = \frac{3\pi}{2} \Rightarrow \ln z = \ln 2 + i\frac{3\pi}{2} \\ -\pi \le \theta < \pi : & \theta = -\frac{\pi}{2} \Rightarrow \ln z = \ln 2 - i\pi/2 \end{cases}$$

complex powers & roots:

examples: $1^2 = 1$

$1^{1/2} = \pm 1$

$1^{1/3} = \ldots ?$

$1^{i} = \ldots ???$

$f(z) = z^\alpha = (e^{\ln z})^\alpha = e^{\alpha \ln z}$ can be multivalued from log.

$= e^{\alpha(\ln r + i\theta)} e^{2\pi i n \alpha}$ $\quad n = 0, \pm 1, \pm 2, \ldots$

• α = integer $\rightarrow$ $e^{2\pi i n \alpha} = +1$ for any $n \Rightarrow$ single-valued.

• α = rational number = $\underbrace{m}_{\text{integer}} + \underbrace{p/q}_{\text{remainder}}$, ~~p≤q~~ $0 < p < q$.

$$e^{2\pi i n \alpha} = 1 \cdot e^{2\pi i n p/q} = \underbrace{\underbrace{1}_{n=0}, \underbrace{e^{2\pi i p/q}}_{n=1}, \ldots,}_{f(z) \text{ is } q\text{-valued function.}} \underbrace{e^{2\pi i q p/q}}_{n=q} = 1$$

• α = irrational or complex. $e^{2\pi i n \alpha}$ = unique for all n

$\rightarrow$ Infinite-valued.

e.g. $f(z) = z^{1/3}$ is 3-valued

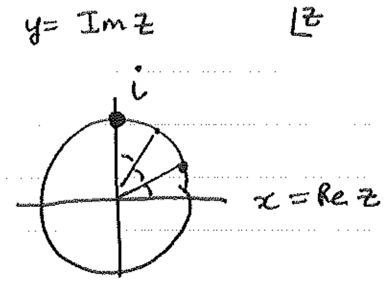
$$z = 1 \rightarrow 1^{1/3} = \underbrace{e^{\frac{1}{3}(\ln 1 + i \cdot 0)}}_{1} e^{2\pi i n/3}$$

$$= 1, e^{2\pi i/3}, e^{4\pi i/3}$$

$$z = i \rightarrow i^{1/3} = e^{\frac{1}{3}(\ln 1 + i\pi/2)} e^{2\pi i n/3}$$

$$= e^{i\pi/6} \cdot 1, e^{i\pi/6 + 2\pi i/3}, e^{i\pi/6 + 4\pi i/3}$$

$$= e^{i\pi/6}, e^{5\pi i/6}, e^{9\pi i/6}$$



check: $(e^{\pi i/6})^3 = e^{3\pi i/6} = e^{\pi i/2} = i$

$(e^{5\pi i/6})^3 = e^{15\pi i/6} = e^{\pi i/2} = i$

$(e^{9\pi i/6})^3 = e^{27\pi i/6} = e^{\pi i/2} = i$

e.g. $1^i = e^{i(\ln 1 + i\cdot 0)} \, e^{2\pi i n \cdot i} = e^{-2\pi n}$, $n = 0, \pm 1, \pm 2, \ldots$

infinitely-many valued.

example: what is $z = (1+i)^{1/2}$? Solve for $z = re^{i\theta}$.

$$z^2 = 1 + i$$

$$r^2 = |z^2| = \sqrt{1+1} = \sqrt{2} \Rightarrow r = |z| = \sqrt[4]{2}$$

$$2\theta = \arg(z^2) = 2\arg(z) = \frac{\pi}{4} + 2\pi i n = 2\theta$$

$$\theta = \arg(z) = \frac{\pi}{8} + 2\pi i n = \frac{\pi}{8}, \frac{9\pi}{8}$$

$$z = \sqrt[4]{2}\, e^{i\pi/8}, \ \sqrt[4]{2}\, e^{9\pi i/8}$$