Derivatives of complex functions

We want to consider what it means to take the derivative of a complex function $f(z)$.

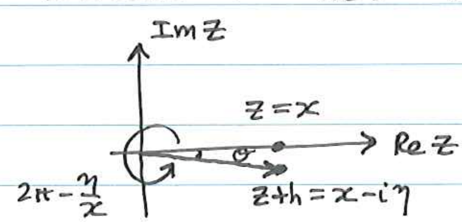
A function $f(z)$ is continuous at z if $\lim_{h\to 0} f(z+h) = f(z)$

example: $f(z) = z^2$

$$\lim_{h\to 0} f(z+h) = \lim_{h\to 0} (z+h)^2 = \lim_{h\to 0} z^2 + 2hz + h^2 = z^2 = f(z)$$

So $f(z) = z^2$ is continuous everywhere (all z)

example: $f(z) = \ln z$, restricted to principle branch with $0 \leq \arg z < 2\pi$. Consider z on the positive real axis and h along the negative imaginary direction. i.e. $z = x$, $h = -i\eta$ $(\eta > 0)$



For small η, $\theta \approx \frac{\eta}{x}$

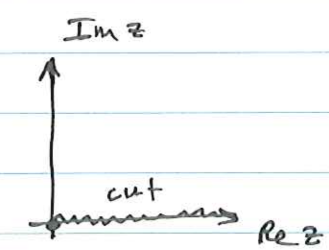
$$f(z) = \ln z = \ln x$$

$$f(z+h) = \ln|z+h| + i\arg(z+h)$$

$$= \ln|x - i\eta| - i\frac{\eta}{x} + 2\pi i$$

$$\lim_{\eta\to 0} f(z+h) = \ln x + 2\pi i \neq f(z) = \ln x$$

The principle branch of $f(z) = \ln z$ is not continuous across the positive real axis. This is a branch cut.



The end of the branch cut at $z=0$ is the branch point.

Location of the branch cut depends on the assumed range for $\arg(z)$. i.e. taking $-\pi \leq \arg z < \pi$, we would have a branch cut along the negative real axis.

The derivative of a complex function is

$$f'(z) = \frac{df}{dz} = \lim_{h\to 0} \frac{f(z+h) - f(z)}{h}$$

A function is <u>differentiable</u> at z only if $f'(z)$ exists (i.e. not ∞) and if $f'(z)$ is independent of the argument of h (i.e. ~~f(z+h) must approach~~ $z+h$ can approach z along any direction in the complex plane and must give same result).

Only a function that is continuous at z is differentiable at z.

Differentiability puts a strong constraint on the form $f(z)$ can take. Consider two cases: h purely real and h purely imaginary. Also recall: $f(z) = u(x,y) + iv(x,y)$

h real case:

$$f'(z) = \lim_{h\to 0} \frac{f(z+h) - f(z)}{h} = \lim_{h\to 0} \frac{u(x+h,y) - u(x,y)}{h} + i\,\frac{v(x+h,y) - v(x,y)}{h}$$

$$= \left(\frac{\partial u}{\partial x}\right) + i\left(\frac{\partial v}{\partial x}\right)$$

h imaginary case:

$$f'(z) = \lim_{h\to 0} \frac{f(z+ih) - f(z)}{ih} = \lim_{h\to 0} \frac{u(x,y+h) - u(x,y)}{ih} + i\,\frac{v(x,y+h) - v(x,y)}{ih}$$

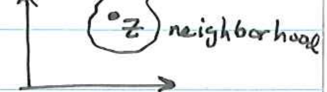
$$= -i\left(\frac{\partial u}{\partial y}\right) + \left(\frac{\partial v}{\partial y}\right) = \left(\frac{\partial v}{\partial y}\right) - i\left(\frac{\partial u}{\partial y}\right)$$

Since the two cases must give the same result, a complex function $f(z) = u(x,y) + i\,v(x,y)$ is differentiable at z if it satisfies the Cauchy-Riemann relations:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

The derivative of a complex function is $\frac{df}{dz} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}$.

A function $f(z)$ is <u>analytic</u> at z if it is differentiable at z <u>and</u> within a small neighborhood of z.



A function that is analytic for all z is called <u>entire</u>.

Analytic complex functions satisfy all the usual rules for computing derivatives of real functions.

example: $f(z) = c = \text{const} \Rightarrow \frac{\partial u}{\partial x} = \frac{\partial v}{\partial x} = 0 \Rightarrow \frac{df}{dz} = 0$

$f(z) = z = x + iy \Rightarrow u = x,\ v = y \Rightarrow \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = 1 \Rightarrow \frac{df}{dz} = 1$

example: product rule: $\frac{d}{dz}(f_1(z) f_2(z)) = f_1'(z) f_2(z) + f_1(z) f_2'(z)$

proof: write $f_1 = u_1 + iv_1$ and $f_2 = u_2 + iv_2$

then $f_1 f_2 = (u_1 + iv_1)(u_2 + iv_2) = (u_1u_2 - v_1v_2) + i(u_1v_2 + u_2v_1)$

$$\frac{d}{dz}(f_1 f_2) = \frac{\partial}{\partial x}(u_1u_2 - v_1v_2) + i\frac{\partial}{\partial x}(u_1v_2 + u_2v_1)$$

$$= (u_1 + iv_1)\left(\frac{\partial u_2}{\partial x} + i\frac{\partial v_2}{\partial x}\right) + \left(\frac{\partial u_1}{\partial x} + i\frac{\partial v_1}{\partial x}\right)(u_2 + iv_2)$$

$$= f_1 f_2' + f_1' f_2$$

example: power rule $\frac{d}{dz}(z^n) = n z^{n-1}$

proof (by induction): already showed the $n=1$ case $\frac{d}{dz}(z) = 1$

Then: if $\frac{d}{dz}(z^{n-1}) = (n-1) z^{n-2}$, then

$$\frac{d}{dz}(z^n) = \frac{d}{dz}(z\, z^{n-1}) = 1\, z^{n-1} + z(n-1) z^{n-2}$$

$= n z^{n-1}$ using product rule

$\cancel{= (n-1)\, z^n}$

So if its true for $n=1$, must be true for $n=2$; if true for $n=2$, must be true for $n=3$; etc. $\Rightarrow$ true for all n

example: $f(z) = z^* = x - iy$

$\Rightarrow u = x,\ v = -y \Rightarrow \dfrac{\partial u}{\partial x} = 1$ and $\dfrac{\partial v}{\partial y} = -1$

Since $\dfrac{\partial u}{\partial x} \neq \dfrac{\partial v}{\partial y}$, $f(z) = z^*$ is not an analytic function (although it is continuous)

An analytic function must be able to be expressed as a function of z only, and not z^*.

proof: write $f(z) = f(x,y) = u + iv$

Since $x = \frac{1}{2}(z + z^*)$ and $y = \frac{-i}{2}(z - z^*)$,

we can consider $x(z,z^*)$, $y(z,z^*)$ as functions of z, z^*.

Using the chain rule:

$$\frac{\partial f}{\partial z^*} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial z^*} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial z^*}$$

$$= \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)\frac{1}{2} + \left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right)\left(\frac{+i}{2}\right)$$

$$= \frac{1}{2}\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) + i\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) = 0$$

by Cauchy-Riemann relations.

So, the following are not analytic:

$f(z) = z^*$, $f(z) = |z| = \sqrt{z z^*}$

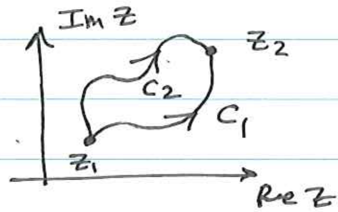
$f(z) = \mathrm{Re}(z) = \frac{1}{2}(z + z^*)$, $f(z) = \mathrm{Im}(z) = \frac{-i}{2}(z - z^*)$

$f(z) = \arg(z) = \arctan\left(\dfrac{\mathrm{Im}(z)}{\mathrm{Re}(z)}\right)$

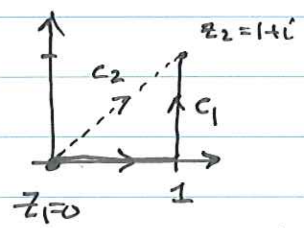
Integrals of complex functions

$$\int_{z_1}^{z_2} dz\, f(z) = \int_{z_1}^{z_2} (dx + i\,dy)\,\big(u(x,y) + i\,v(x,y)\big)$$

Since z_1 and z_2 are points in the complex plane, need to also specify the path from z_1 to z_2 (e.g. C_1 or C_2)



example: consider $f(z) = z$, integrated from $z_1 = 0$ to $z_2 = 1+i$ along two different paths C_1 and C_2



$$\int_{C_1} dz\, f(z) = \int_0^1 dz\, f(x, y=0) + \int_1^{1+i} dz\, f(x=1, y)$$

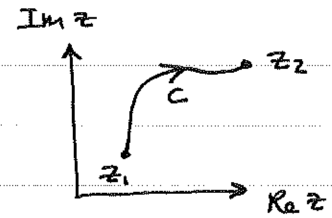
$$= \int_0^1 dx\,(x + i\cdot 0) + \int_0^1 i\,dy\,(1 + iy)$$

$$= \frac{1}{2}x^2\Big|_0^1 + i\left(y + \frac{i}{2}y^2\right)\Big|_0^1 = \frac{1}{2} + i - \frac{1}{2} = i$$

$$\int_{C_2} dz\, f(z) = \int_0^{1+i} (dx + i\,dy)(x + iy) = \int_0^1 (dx + i\,dx)(x + ix) =$$

$$= \int_0^1 dx\; x\,(1+i)^2 = \frac{1}{2}(1+i)^2 = \frac{1}{2}(1 + 2i - 1) = i$$

Note: same ~~[illegible]~~ result for both C_1 and C_2.



In specifying a path C, you also need to specify the ~~direction~~ orientation, i.e. which is the start (z_1) and which is the end (z_2).

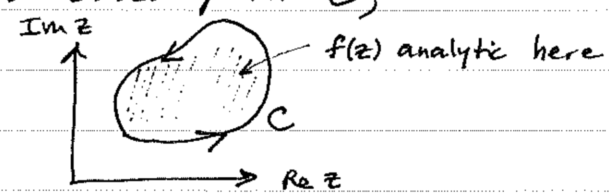
Reversing the ~~direction~~ orientation of the path ($C \to -C$), flips the bounds of integration ($\int_{z_1}^{z_2} \to \int_{z_2}^{z_1}$), which flips the sign of the integral.

$$\int_C dz\, f(z) = \int_{z_1}^{z_2} dz\, f(z) = -\int_{z_2}^{z_1} dz\, f(z) = -\int_{-C} dz\, f(z)$$

Denote "$-C$" the path C in the reverse direction.
Convention: all closed paths are oriented in the counterclockwise direction.

Cauchy's theorem: One of the most important results in complex analysis. It states: for a closed path C,

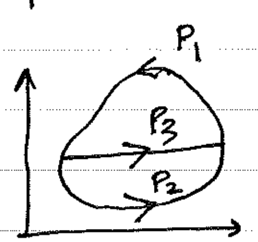
$$\oint_C dz\, f(z) = 0$$



if $f(z)$ is analytic for all z on and within C.

Note: "$\oint$" is an integral along a closed loop C that has the same start & end point.

proof: First, consider dividing C into two parts, defined by the paths P_1, P_2, P_3 as follows:



$C = P_1 + P_2$
Can also define two smaller contours
$C_1 = P_1 + P_3$ and $C_2 = P_2 - P_3$

Then $\oint_C dz\, f(z) = \int_{P_1} dz\, f(z) + \int_{P_2} dz\, f(z)$

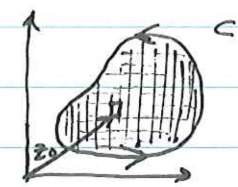
$$= \int_{P_1} dz\, f(z) + \int_{P_2} dz\, f(z) + \int_{P_3} dz\, f(z) + \int_{-P_3} dz\, f(z)$$

$$= \oint_{C_1} dz\, f(z) + \oint_{C_2} dz\, f(z)$$

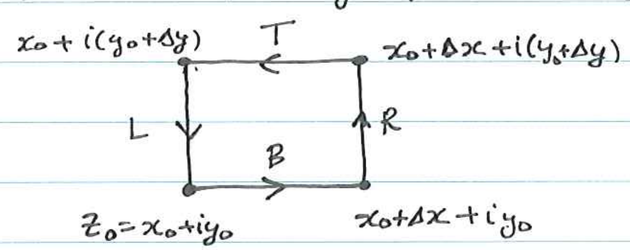
So

$$\oint_C dz\, f(z) \quad = \quad \oint_{C_1} dz\, f(z) + \oint_{C_2} dz\, f(z)$$

We can divide C into an arbitrary number of smaller closed loops $C_1, \dots, C_N$; and $\oint_C dz\, f(z) = \sum_{i=1}^{N} \oint_{C_i} dz\, f(z)$. If C is oriented counter-clockwise, all subloops C_i are also oriented counter-clockwise.



Now, consided dividing C into a large number of infinitesimal rectangles of size Δx by Δy. Consider the rectangle C_i located at $z = z_0$



Evaluate $\oint_{C_i} dz\, f(z) = \left(\int_B + \int_R + \int_T + \int_L\right) dz\, f(z)$ for this rectangle C_i

$$\left(\int_B + \int_T\right) dz\, f(z) = \int_{x_0}^{x_0+\Delta x} dx\, \left(u(x,y_0) + i\, v(x,y_0)\right) + \int_{x_0+\Delta x}^{x_0} dx\, \left(u(x,y_0+\Delta y) + i\, v(x,y_0+\Delta y)\right)$$

$$= \int_{x_0}^{x_0+\Delta x} dx \Big\{ \left(u(x,y_0) + i v(x,y_0)\right) - \left(u(x,y_0) + i\, v(x,y_0)\right) - \left(\frac{\partial u}{\partial y} + i \frac{\partial v}{\partial y}\right)\Delta y \Big\}$$ Taylor expand

$$= -\int_{x_0}^{x_0+\Delta x} dx\, \left(\frac{\partial u}{\partial y} + i \frac{\partial v}{\partial y}\right)\Delta y = -\left(\frac{\partial u}{\partial y} + i \frac{\partial v}{\partial y}\right)\Delta x\, \Delta y$$

$$\left(\int_R + \int_L\right) dz\, f(z) = \int_{y_0}^{y_0+\Delta y} i\,dy\, \left(u(x_0+\Delta x, y) + i v(x_0+\Delta x, y)\right) + \int_{y_0+\Delta y}^{y_0} i\,dy\, \left(u(x_0, y_0) + i v(x_0, y_0)\right)$$

$$= i\int_{y_0}^{y_0+\Delta y} dy\, \left\{ \left(u(x_0, y_0) + i v(x_0, y)\right) + \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)\Delta x - \left(u(x_0, y) + i v(x_0, y)\right) \right\}$$

$$= i\int_{y_0}^{y_0+\Delta y} dy\, \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)\Delta x = i\left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)\Delta x\, \Delta y$$

The sum is: $\oint_{C_i} dz\, f(z) = \left\{ -\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} - i\frac{\partial v}{\partial y} + i\frac{\partial u}{\partial x} \right\} \Delta x\, \Delta y = 0$

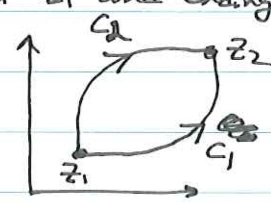
by Cauchy - Riemann relations $\left(\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y},\ \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}\right)$

Since the integral of any tiny rectangle is zero: $\oint_{C_i} dz\, f(z) = 0$
then the sum is $\oint_C dz\, f(z) = \sum_{i=1}^{N} \oint_{C_i} dz\, f(z) = 0$.

This holds provided Cauchy - Riemann relations are satisfied everywhere on and within C, so $f(z)$ must be analytic on and within C.

Cauchy's theorem implies:

(1) For any two paths C_1 and C_2 both starting at z_1 and ending at z_2, $\int_{C_1} dz\, f(z) = \int_{C_2} dz\, f(z)$ if $f(z)$ is analytic on C_1 and C_2 and within the enclosed region.



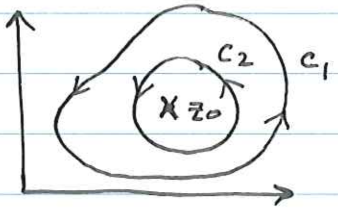
proof: $0 = \oint_{C_1 - C_2} dz\, f(z) = \int_{C_1} dz\, f(z) - \int_{C_2} dz\, f(z)$

$$\Rightarrow \int_{C_1} dz\, f(z) = \int_{C_2} dz\, f(z)$$

(2) For any two <u>closed</u> paths C_1 and C_2, if C_1 and C_2 can be <u>deformed</u> into one another without passing through any regions where $f(z)$ is not analytic, then

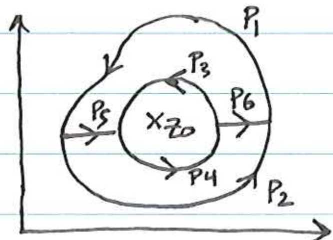
$$\oint_{C_1} dz\, f(z) = \oint_{C_2} dz\, f(z)$$

e.g. Suppose $f(z)$ is not analytic at $z = z_0$. Then



$\oint_{C_1} dz\, f(z) = \oint_{C_2} dz\, f(z)$, but note $\oint_{C_1} dz\, f(z) \neq 0$ in general since not analytic inside C_1.

proof: Consider the following series of paths:



$$\oint_{C_1} dz\, f(z) = \int_{P_1} dz\, f(z) + \int_{P_2} dz\, f(z)$$

$$\oint_{C_2} dz\, f(z) = \int_{P_3} dz\, f(z) + \int_{P_4} dz\, f(z)$$

Cauchy's theorem implies:

$$\int_{P_1} dz\, f(z) + \int_{P_5} dz\, f(z) + \int_{-P_3} dz\, f(z) + \int_{P_6} dz\, f(z) = 0$$

$$\int_{P_2} dz\, f(z) + \int_{-P_6} dz\, f(z) + \int_{-P_4} dz\, f(z) + \int_{-P_5} dz\, f(z) = 0$$

Adding them together:

$$\int_{P_1} dz\, f(z) + \int_{P_2} dz\, f(z) - \int_{P_3} dz\, f(z) - \int_{P_4} dz\, f(z) = 0$$

$$\oint_{C_1} dz\, f(z) = \oint_{C_2} dz\, f(z)$$