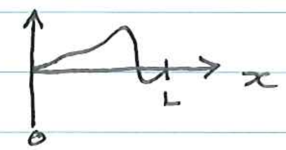
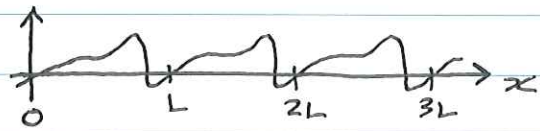
III. Fourier Series

Fourier series is a useful way of representing a function $f(x)$ that is either (1) only defined within a finite domain (e.g. $0 \le x \le L$)
or (2) periodic in x (say, with period L)

case (1): 

$f(x)$ only defined for $0 \le x \le L$

case (2): 

$f(x)$ is periodic.
$f(x+L) = f(x)$
$f(x+nL) = f(x)$ for $n = 0, \pm 1, \pm 2, ..$

Fourier's theorem: Any (sufficiently well-behaved) function $f(x)$ defined in the domain $0 \le x \le 2\pi$ may be represented by the sum of sines & cosines:

$$f(x) = \sum_{n=0}^{\infty} \left(a_n \sin(nx) + b_n \cos(nx) \right)$$

where a_n, b_n are constants.
(note: a_0 is irrelevant since $\sin(0) = 0$)
and also $b_0 \cos(0) = b_0$ $\rightarrow f(x) = b_0 + \sum_{n=1}^{\infty} (a_n \sin nx + b_n \cos nx)$

Equivalently, $f(x)$ may instead be represented by a sum of complex exponentials:

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}$$

$$= \sum_{n=-\infty}^{\infty} c_n \left(\cos(nx) + i \sin(nx) \right)$$

$$= \sum_{n=0}^{\infty} c_n \left(\cos(nx) + i \sin(nx) \right) + \sum_{n=-\infty}^{-1} c_n \left(\cos(nx) + i \sin(nx) \right)$$

$$= c_0 + \sum_{n=1}^{\infty} c_n \left(\cos(nx) + i \sin(nx) \right) + \sum_{n=1}^{\infty} c_{-n} \left(\cos(nx) - i \sin(nx) \right)$$

So we have $b_0 = c_0$

$b_n = c_n + c_{-n}$ (for $n = 1, \dots$)

$a_n = i(c_n - c_{-n})$ (for $n = 1, \dots$)

Or equivalently $\left.\begin{array}{l} c_n = \frac{1}{2}(b_n - ia_n) \\ c_{-n} = \frac{1}{2}(b_n + ia_n) \end{array}\right\}$ for $n \geq 1$

The trick is to find the Fourier coefficients a_n, b_n (or c_n) for $f(x)$.

Note that $\sum_{n=0}^{\infty} (a_n \sin(nx) + b_n \cos(nx))$ is also periodic under $x \to x + 2\pi$ since $\sin(n(x+2\pi)) = \sin(nx)$ and $\cos(n(x+2\pi)) = \cos(nx)$. So both case (1) & (2) have the same Fourier series (if $f(x)$ is the same).

Thus we can find the Fourier series for a periodic function $f(x)$ by just considering a single period (e.g. $0 \leq x \leq 2\pi$).

Finding the Fourier coefficients

First evaluate $\int_0^{2\pi} dx\, e^{inx} e^{-imx}$ where n, m are integers

if $n = m$, then $\int_0^{2\pi} dx\, e^{inx} e^{-imx} = \int_0^{2\pi} dx = 2\pi$

if $n \neq m$, then $\int_0^{2\pi} dx\, e^{inx} e^{-imx} = \int_0^{2\pi} dx\, e^{i(n-m)x}$

$$= \frac{1}{i(n-m)} e^{i(n-m)x} \Big|_0^{2\pi} = \frac{1}{i(n-m)} \left\{ e^{2\pi i(n-m)} - 1 \right\} = 0$$

So $\int_0^{2\pi} dx\, e^{inx} e^{-imx} = 2\pi \delta_{mn}$

So e^{inx} (where $n = 0, \pm 1, \pm 2, \dots$) are a basis of orthogonal functions.

Next consider sine & cosine (where $n, m \geq 0$)

$$\int_0^{2\pi} dx \cos(nx)\cos(mx) = \int_0^{2\pi} dx \frac{1}{2}(e^{inx} + e^{-inx})(e^{imx} + e^{-imx})\frac{1}{2}$$

$$= \int_0^{2\pi} dx \frac{1}{4}\left(e^{inx}e^{imx} + e^{inx}e^{-imx} + e^{-inx}e^{imx} + e^{-inx}e^{-imx}\right)$$

$$= \frac{1}{4} 2\pi \left(\delta_{n,-m} + \delta_{nm} + \delta_{nm} + \delta_{n,-m}\right)$$

note: $n,m \geq 0$ so $\delta_{n,-m}$ only nonzero for $n=m=0$.

$$\int_0^{2\pi} dx \cos(nx)\cos(mx) = \begin{cases} 2\pi & \text{for } n=m=0 \\ \pi & \text{for } n=m \geq 1 \\ 0 & \text{for } n \neq m \end{cases}$$

$$\int_0^{2\pi} dx \sin(nx)\sin(mx) = \int_0^{2\pi} dx \frac{1}{2i}(e^{inx} - e^{-inx})\frac{1}{2i}(e^{imx} - e^{-imx})$$

$$= -\frac{1}{4}\int_0^{2\pi} dx \left(e^{inx}e^{imx} - e^{inx}e^{-imx} - e^{-inx}e^{imx} + e^{-inx}e^{-imx}\right)$$

$$= -\frac{1}{4} 2\pi \left(\delta_{n,-m} - \delta_{nm} - \delta_{nm} + \delta_{n,-m}\right) = \pi \delta_{nm}$$

(note: $n,m=0$ case vanishes.

$$\int_0^{2\pi} dx \cos(nx)\sin(mx) = \int_0^{2\pi} dx \frac{1}{2}(e^{inx} + e^{-inx})\frac{1}{2i}(e^{imx} - e^{-imx})$$

$$= \frac{1}{4i}\int_0^{2\pi} dx \left(e^{inx}e^{imx} - e^{inx}e^{-imx} + e^{-inx}e^{imx} - e^{-inx}e^{-imx}\right)$$

$$= \frac{1}{4i} 2\pi \left(\delta_{n,-m} - \delta_{nm} + \delta_{nm} - \delta_{n,-m}\right) = 0$$

So sines & cosines also form a basis of orthogonal functions:

$$\int_0^{2\pi} dx \cos(nx)\cos(mx) = \int_0^{2\pi} dx \sin(nx)\sin(mx) = \pi \delta_{nm} \quad \text{for } n,m \geq 1$$

$$\int_0^{2\pi} dx \cos(nx)\sin(mx) = 0$$

Then we can evaluate a_n, b_n:

$$\int_0^{2\pi} dx \, \sin(mx) f(x) = \int_0^{2\pi} dx \, \sin(mx) \sum_{n=0}^{\infty} \left(a_n \sin(nx) + b_n \cos(nx)\right)$$

$$= \sum_{n=0}^{\infty} a_n \pi \delta_{nm} = a_m \pi$$

$$\int_0^{2\pi} dx \, \cos(mx) f(x) = \int_0^{2\pi} dx \, \cos(mx) \sum_{n=0}^{\infty} \left(a_n \sin(nx) + b_n \cos(nx)\right)$$

$$= \sum_{n=0}^{\infty} b_n \times \begin{cases} 2\pi \, \delta_{nm} & (m=0) \\ \pi \, \delta_{nm} & (m \geq 1) \end{cases} = b_m \times \begin{cases} 2\pi & \text{if } m=0 \\ \pi & \text{if } m \geq 1 \end{cases}$$

So
$$\boxed{\begin{aligned} a_m &= \frac{1}{\pi} \int_0^{2\pi} dx \, \sin(mx) f(x) \\ b_m &= \frac{1}{\pi} \int_0^{2\pi} dx \, \cos(mx) f(x) \end{aligned} \Bigg\} \; m \geq 1 \\ b_0 = \frac{1}{2\pi} \int_0^{2\pi} dx \, f(x)}$$

Or in terms of c_n's:

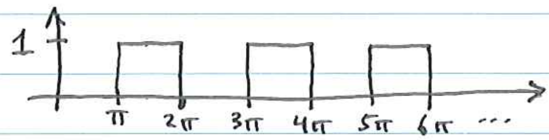
$$c_m = \frac{1}{2}(b_m - i a_m) = \frac{1}{2\pi} \int_0^{2\pi} dx \, e^{-imx} f(x) \qquad \text{for } m \geq 1$$

$$c_{-m} = \frac{1}{2}(b_m + i a_m) = \frac{1}{2\pi} \int_0^{2\pi} dx \, e^{imx} f(x) \qquad \text{for } m \geq 1$$

$$c_0 = b_0 = \frac{1}{2\pi} \int_0^{2\pi} dx \, f(x) \qquad m=0$$

Equivalent to a single formula: $\boxed{c_m = \frac{1}{2\pi} \int_0^{2\pi} dx \, e^{-imx} f(x)}$ for all m.

example: find the Fourier series for a square wave



Suffices to consider a single period $0 \le x \le 2\pi$.

so $$f(x) = \begin{cases} 0 & 0 \le x \le \pi \\ 1 & \pi \le x \le 2\pi \end{cases}$$

Compute Fourier coefficients:

$$b_0 = \frac{1}{2\pi}\int_0^{2\pi} dx\, f(x) = \frac{1}{2\pi}\int_\pi^{2\pi} dx \cdot 1 = \frac{1}{2}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} dx\, \cos(nx)\, f(x) = \frac{1}{\pi}\int_\pi^{2\pi} dx\, \cos(nx)$$

$$= \frac{1}{\pi}\frac{1}{n}\sin(nx)\Big|_\pi^{2\pi} = 0 \qquad \text{for } n \ge 1$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} dx\, \sin(nx)\, f(x) = \frac{1}{\pi}\int_\pi^{2\pi} dx\, \sin(nx)$$

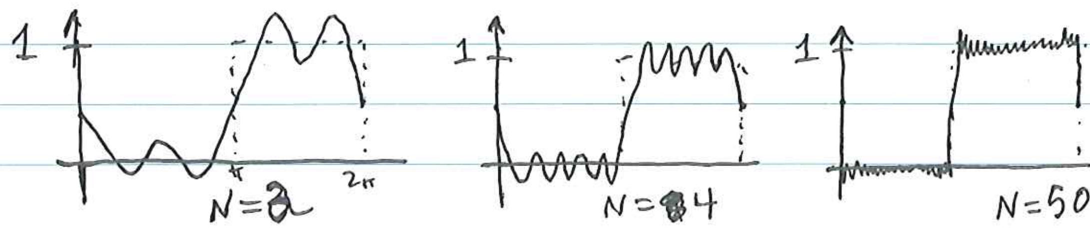
$$= \frac{1}{n\pi}\left(-\cos(nx)\right)\Big|_\pi^{2\pi} = \frac{-1}{n\pi}\left(1 - (-1)^n\right)$$

So $$f(x) = \frac{1}{2} - \sum_{n=1}^{\infty} \frac{1}{n\pi}(1-(-1)^n)\sin(nx)$$

$$= \frac{1}{2} - \frac{2}{\pi}\left(\sin(x) + \frac{1}{3}\sin(3x) + \frac{1}{5}\sin(5x) + \dots \right)$$

Although an infinite number of terms is needed to reproduce $f(x)$, we can consider the first N terms in the series:

$$f_N(x) = \frac{1}{2} - \frac{2}{\pi}\left(\sin x + \frac{1}{3}\sin(3x) + \dots + \frac{1}{2N+1}\sin((2N+1)x) \right)$$



Fourier Series of a function $f(x)$ over an interval $0 \le x \le L$.

$$f(x) = \sum_{n=0}^{\infty} \left(a_n \sin\left(\frac{2\pi n x}{L}\right) + b_n \cos\left(\frac{2\pi n x}{L}\right) \right)$$

$$= \sum_{n=-\infty}^{\infty} C_n \exp\left(i \frac{2\pi n x}{L}\right)$$

Similar to expanding a vector $\vec{V} = \sum_{i=1}^{N} v_i \hat{e}_i$ in terms of components v_i and basis vectors $\hat{e}_i$.

Basis functions: $\left\{ \begin{matrix} \sin\left(\frac{2\pi n x}{L}\right) & n \ge 1 \\ \cos\left(\frac{2\pi n x}{L}\right) & n \ge 0 \end{matrix} \right\}$ or $\left\{ \exp\left(i \frac{2\pi n x}{L}\right) \quad n = 0, \pm 1, \ldots \right\}$

are like the basis vectors $\hat{e}_i$. (Infinite dimensional vector space.)

~~Basis vectors are othogonal~~:

Basis functions are orthogonal: (like $\langle \hat{e}_i, \hat{e}_j \rangle = \delta_{ij}$)

$$\frac{2}{L} \int_0^L dx \, \sin\left(\frac{2\pi n x}{L}\right) \sin\left(\frac{2\pi m x}{L}\right) = \frac{2}{L} \int_0^L dx \, \cos\left(\frac{2\pi n x}{L}\right) \cos\left(\frac{2\pi m x}{L}\right) = \delta_{nm} \quad (n \ge 1)$$

$$\frac{2}{L} \int_0^L dx \, \sin\left(\frac{2\pi n x}{L}\right) \cos\left(\frac{2\pi m x}{L}\right) = 0$$

or

$$\frac{1}{L} \int_0^L dx \, e^{i 2\pi n x / L} \, e^{-i 2\pi m x / L} = \delta_{nm}$$

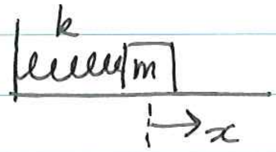
Fourier coefficients: (similar to $v_i = \langle \hat{e}_i, \vec{V} \rangle$)

$$a_n = \frac{2}{L} \int_0^L dx \, f(x) \sin\left(\frac{2 n \pi x}{L}\right) \qquad b_n = \frac{2}{L} \int_0^L dx \, f(x) \cos\left(\frac{2 n \pi x}{L}\right) \quad (n \ge 1)$$

$$b_0 = \frac{1}{L} \int_0^L dx \, f(x)$$

or $\quad C_n = \frac{1}{L}\int_0^L dx\, f(x)\, e^{-i2\pi n x/L} \qquad (n = 0, \pm 1, \pm 2, \ldots)$

<u>Application: Forced harmonic oscillator</u>



$$\ddot{x} + \gamma \dot{x} + \omega^2 x = F(t)$$

$\omega^2 = k/m$

γ is damping coefficient (e.g. air resistance)

Suppose $F(t)$ is a periodic function (with period τ).
Look for solutions for $x(t)$ that are also periodic.

Fourier expand: $\quad x(t) = \sum_{n=-\infty}^{\infty} c_n e^{i2\pi n t/\tau}$

$$F(t) = \sum_{n=-\infty}^{\infty} f_n e^{i2\pi n t/\tau}$$

Plug in to diff. eqn:

$$\sum_{n=-\infty}^{\infty} \left[-\left(\frac{2\pi n}{\tau}\right)^2 + \gamma i \left(\frac{2\pi n}{\tau}\right) + \omega^2\right] c_n e^{i2\pi n t/\tau} = \sum_{n=-\infty}^{\infty} f_n e^{i2\pi n t/\tau}$$

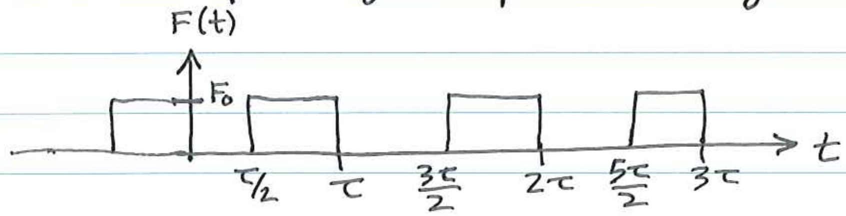
Act on both sides with $\frac{1}{\tau}\int_0^\tau dt\, e^{-i2\pi m t/\tau}$ and use orthogonality.

$$\left[-\left(\frac{2\pi m}{\tau}\right)^2 + \gamma i \left(\frac{2\pi m}{\tau}\right) + \omega^2\right] c_m = f_m$$

So $\quad c_m = \dfrac{f_m}{\omega^2 - \omega_m^2 + i\gamma\omega_m} \qquad$ (where $\omega_n = \frac{2\pi n}{\tau}$)

We can compute c_m from $f_m = \frac{1}{\tau}\int_0^\tau dt\, F(t)\, e^{-i2\pi m t/\tau}$

example: Square pulse driving force



so $\quad F(t) = \begin{cases} 0 & 0 \le t < \tau/2 \\ F_0 & \tau/2 \le t < \tau \end{cases}$ $\qquad$ etc.

$$f_n = \frac{1}{\tau}\int_0^\tau dt\ F_0\ \theta(\tau - t) \times e^{-i2\pi n t/\tau}$$

$$= \frac{F_0}{\tau}\int_{\tau/2}^{\tau} dt\ e^{-2\pi i n t/\tau} = \frac{F_0}{\tau}\frac{\tau}{-2\pi i n}\left(e^{-2\pi i n} - e^{-\pi i n}\right)$$

$$= \frac{F_0}{-2\pi i n}\left(1 - e^{-in\pi}\right) = \begin{cases} \dfrac{iF_0}{\pi n} & n = \text{odd} \\ 0 & n = \text{even} \end{cases}$$

So $$x(t) = \sum_{\substack{n=-\infty \\ n \text{ odd}}}^{\infty} \frac{iF_0}{\pi n} \cdot e^{i2\pi n t/\tau} \frac{1}{\omega^2 - \omega_n^2 + i\gamma\omega_n}$$

$$= \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{iF_0}{\pi n} e^{i\omega_n t} \frac{\omega^2 - \omega_n^2 - i\gamma\omega_n}{(\omega^2-\omega_n^2)^2 + \gamma^2\omega_n^2}$$

$$+ \sum_{\substack{n=1 \\ \text{odd}}}^{\infty} \frac{-iF_0}{\pi n} e^{-i\omega_n t} \frac{\omega^2 - \omega_n^2 + i\gamma\omega_n}{(\omega^2-\omega_n^2)^2 + \gamma^2\omega_n^2}$$

$$= \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{2F_0}{\pi n} \frac{\gamma\omega_n \cos(\omega_n t) - (\omega^2 - \omega_n^2)\sin(\omega_n t)}{(\omega^2-\omega_n^2)^2 + \gamma^2\omega_n^2}$$

What about the homogeneous solution?

$$\ddot{x} + \gamma\dot{x} + \omega^2 x = 0$$

Solution: $\quad x(t) = A e^{i\lambda_+ t} + B e^{-i\lambda_- t}$

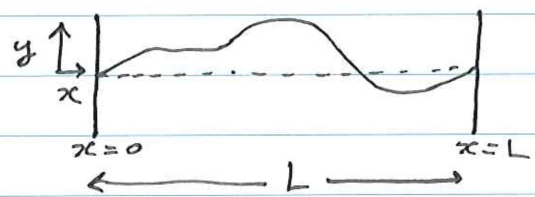
where $\quad \lambda_\pm = \frac{i\gamma}{2} \pm \frac{1}{2}\sqrt{4\omega^2 - \gamma^2}$

Can add this to the inhomogeneous solution, but decays away at late times.

~~example~~

Application: plucked string

Consider a string of length L, fixed at the ends, with displacement $y(x,t)$

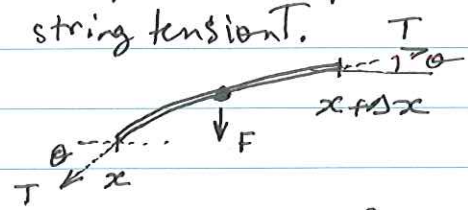


$$y(0,t) = y(L,t) = 0 .$$

$y(x,t)$ satisfies the wave equation $\dfrac{\partial^2 y}{\partial t^2} - v^2 \dfrac{\partial^2 y}{\partial x^2} = 0 .$

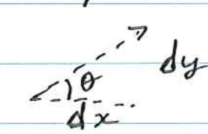
Wave equation:

Consider a tiny element of the string of mass Δm and length Δx, and string tension T.



Newton's 2nd law: $\Delta m \dfrac{\partial^2 y}{\partial t^2} = T\left(\sin\theta(x+\Delta x) - \sin\theta(x)\right)$

In the small angle limit, $\sin\theta(x) \simeq \theta(x) \approx \dfrac{\partial y}{\partial x}(x)$



So $\Delta m \dfrac{\partial^2 y}{\partial t^2} = T \dfrac{\partial^2 y}{\partial x^2}\Delta x .$

Linear density of string $= \rho = \dfrac{\Delta m}{\Delta x} = \dfrac{\text{mass}}{\text{length}}$

$$\Rightarrow \frac{\partial^2 y}{\partial t^2} = v^2 \frac{\partial^2 y}{\partial x^2} \quad \text{where} \quad v = \sqrt{T/\rho} = \text{wave velocity.}$$

For fixed t, can expand $y(x,t)$ in Fourier series of sine & cosine with period $2L$.

$$y(x,t) = \sum_{n=0}^{\infty} \left(a_n \sin\left(\frac{\pi n x}{L}\right) + b_n \cos\left(\frac{n\pi x}{L}\right) \right)$$

$$y(0,t) = y(L,t) = 0 \Rightarrow b_n = 0$$

So $y(x,t) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{\pi n x}{L}\right)$

time dependence: $a_n = a_n(t)$ is function of time.

Plug in to wave equation:

$$\frac{\partial^2 y}{\partial t^2} = \sum_{n=1}^{\infty} \ddot{a}_n(t) \sin\left(\frac{\pi n x}{L}\right) = -v^2 \sum_{n=1}^{\infty} a_n(t) \sin\left(\frac{\pi n x}{L}\right) \left(\frac{\pi n}{L}\right)^2$$

Act on both sides with $\frac{2}{L}\int_0^L dx \ \sin\left(\frac{\pi m x}{L}\right)$:

$$\cancel{\ddot{a}_m = v^2 a_m \Rightarrow a_m(t) = A_m}$$

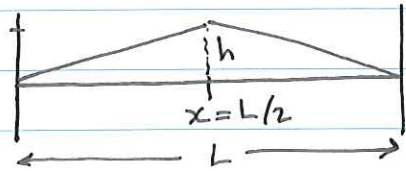
$$\ddot{a}_m = -\frac{v^2 \pi^2 m^2}{L^2} a_m$$

$$a_m(t) = A_m \sin(\omega_m t) + B_m \cos(\omega_m t)$$

where $\omega_m = \frac{v\pi m}{L}$

So $$y(x,t) = \sum_{n=1}^{\infty} \left(A_n \sin(\omega_n t) + B_n \cos(\omega_n t) \right) \cdot \sin\left(\frac{\pi n x}{L}\right)$$

A_n, B_n fixed by initial conditions.



String displaced at $x = L/2$ by $y(L/2, 0) = h$.

Initial condition:

$$y(x,0) = \begin{cases} \frac{2hx}{L} & 0 \leq x \leq L/2 \\ \frac{2h}{L}(L-x) & \frac{L}{2} \leq x \leq L \end{cases}$$

$$\dot{y}(x,0) = 0 \quad \rightarrow \quad A_n = 0$$

$$y(x,0) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{\pi n x}{L}\right) = \begin{cases} 2hx/L & 0 \leq x \leq L/2 \\ \frac{2h}{L}(L-x) & \frac{L}{2} \leq x \leq L \end{cases}$$

Act with $\frac{2}{L}\int_0^L dx \sin\left(\frac{\pi m x}{L}\right)$

$$B_m = \frac{2}{L}\int_0^{L/2} dx \, \frac{2hx}{L} \sin\left(\frac{\pi m x}{L}\right) + \frac{2}{L}\int_{L/2}^{L} dx \, \frac{2h}{L}(L-x) \sin\left(\frac{\pi m x}{L}\right)$$

change of variables: $x' = \frac{x}{L}$

$$B_m = 4h \int_0^{1/2} dx' \, x' \sin(\pi m x') + 4h \int_{1/2}^{1} dx' \, (1-x') \sin(\pi m x')$$

$$= \frac{8h}{m^2\pi^2} \sin\left(\frac{m\pi}{2}\right)$$ only odd m is nonvanishing

So $$y(x,t) = \sum_{n=1}^{\infty} \frac{8h}{n^2\pi^2} \sin\left(\frac{n\pi}{2}\right) \cos(\omega_n t) \sin\left(\frac{\pi n x}{L}\right)$$

If you pluck the string at $x = L/k$ (here $k = 2$) no modes that are multiples of k appear. (here, only odd modes)