# Vector spaces, eigenvalue problems, & group theory

## Review: Linear algebra basics

Vectors $\vec{u}, \vec{v}, \vec{w}, \ldots$ are objects making up a vector space $V$ with the properties:

(1) closure under addition: $\vec{w} = \vec{u} + \vec{v}$ is in $V$ if $\vec{u}, \vec{v}$ are in $V$.

(2) commutative under addition: $\vec{u} + \vec{v} = \vec{v} + \vec{u}$

(3) associative under addition: $(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$

(4) identity under addition: $\vec{u} + \vec{0} = \vec{u}$ where $\vec{0}$ is "zero vector"

(5) inverse under addition: inverse vector $-\vec{u}$ is in $V$, such that $\vec{u} + (-\vec{u}) = \vec{0}$.

Let $a, b, \ldots$ be scalar numbers (real or complex)

(6) closure under multiplication: $\vec{v} = a\vec{u}$ is in $V$ for any $\vec{u}$ in $V$ and any $a$.

(7) Associativity under multiplication: $a(b\vec{u}) = (ab)\vec{u}$

(8) Identity under multiplication: $1 \cdot \vec{u} = \vec{u}$

(9) Distributive properties: $a(\vec{u} + \vec{v}) = a\vec{u} + a\vec{v}$

$$(a+b)\vec{u} = a\vec{u} + b\vec{u}$$

Summary of all the rules: any general linear combination $a\vec{u} + b\vec{v}$ is a vector in $V$ for any $\vec{u}, \vec{v}$ in $V$ and any real or complex scalars $a, b$. (Plus commutative & associative properties of addition and multiplication.)

Real vector space: Scalars $a, b, ..$ must be real.

Complex vector space: scalars $a, b, \ldots$ can be complex.

Basis vectors are a set of $N$ vectors $\vec{e}_1, \ldots, \vec{e}_N$ in $V$ such that any vector in $V$ can be represented as linear combination

$$\vec{v} = \sum_{i=1}^{N} v_i \vec{e}_i \quad \rightarrow \text{represented } \vec{v} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_N \end{pmatrix}$$

$v_i$ are components of $\vec{v}$. Depend on basis.
Are real (complex) for real (complex) vector space.

Inner product: $\langle \vec{u}, \vec{v} \rangle$ = real (complex) scalar

Real vector space: $\langle \vec{u}, \vec{v} \rangle = \vec{u} \cdot \vec{v}$ (dot product)

Rules for inner product:

(1) Additivity: $\langle \vec{u} + \vec{v}, \vec{w} \rangle = \langle \vec{u}, \vec{w} \rangle + \langle \vec{v}, \vec{w} \rangle$

(2) Linear under multiplication: $\langle \vec{u}, a\vec{v} \rangle = a \langle \vec{u}, \vec{v} \rangle$

(3) Positive-definite: $\langle \vec{u}, \vec{u} \rangle \geq 0$ and $\langle \vec{u}, \vec{u} \rangle = 0$ only if $\vec{u} = \vec{0}$ is zero vector

For a <u>real</u> vector space:

(1) $\langle \vec{u}, \vec{v} \rangle = \langle \vec{v}, \vec{u} \rangle =$ real number

(2) $\langle a\vec{u}, \vec{v} \rangle = \langle \vec{u}, a\vec{v} \rangle = a \langle \vec{u}, \vec{v} \rangle$

For a <u>complex</u> vector space:

(1) $\langle \vec{u}, \vec{v} \rangle = \langle \vec{v}, \vec{u} \rangle^* =$ complex number

(2) $\langle a\vec{u}, \vec{v} \rangle = a^* \langle \vec{u}, \vec{v} \rangle \neq a \langle \vec{u}, \vec{v} \rangle = \langle \vec{u}, a\vec{v} \rangle$

These rules are required by positive definiteness.

Proof: Consider vectors $\vec{u}$ and $\vec{v} = i\vec{u}$, assume $\vec{u} \neq \vec{0}$.

Must have $\langle \vec{u}, \vec{u} \rangle > 0$ and $\langle \vec{v}, \vec{v} \rangle > 0$.

$$\langle \vec{v}, \vec{v} \rangle = \langle i\vec{u}, i\vec{u} \rangle = i \langle i\vec{u}, \vec{u} \rangle = \underset{\text{rule}(2)}{i} \underbrace{(-i)}_{i^*} \langle \vec{u}, \vec{u} \rangle = \langle \vec{u}, \vec{u} \rangle$$

$$\text{or} \quad \underset{\text{rule}(1)}{=} i \langle \vec{u}, i\vec{u} \rangle^* = i \underbrace{(-i)}_{i^*} \langle \vec{u}, \vec{u} \rangle = \langle \vec{u}, \vec{u} \rangle$$

if we didn't use rules for complex vector space, then we would get $i^2 = -1$ instead of $i i^* = +1$. $\Rightarrow \langle \vec{u}, \vec{u} \rangle = -\langle \vec{v}, \vec{v} \rangle$ can't both be positive.

~~Linearly Independent Basis Vectors~~

Assume basis vectors satisfy following properties:

(1) linearly independent
(2) span the vector space
(3) orthogonal
(4) normalized to unity (unit vector)

(3) and (4): orthonormal

Orthonormal basis: $\hat{e}_i$ satisfies (hat = unit vector)

$$\langle \hat{e}_i, \hat{e}_j \rangle = \delta_{ij}$$

Number of linearly independent basis vectors that span the vector space = dimension of vector space.

Example: 3-dimensional space $\mathbb{R}^3$
3 basis vectors $\{\hat{e}_x, \hat{e}_y, \hat{e}_z\}$

basis e.g. $\{\hat{e}_x, \hat{e}_y\}$ doesn't span $\mathbb{R}^3$ since $\vec{v}$ pointing in $\hat{e}_z$ direction cannot be expressed as linear combination of $a\hat{e}_x + b\hat{e}_y$.

basis e.g. $\left\{\hat{e}_x, \hat{e}_y, \hat{e}_z, \hat{e}_4 = \frac{\hat{e}_x + \hat{e}_y}{\sqrt{2}}\right\}$ not linearly independent since $\hat{e}_4$ is linear comb. of other basis vectors.

Inner product in terms of components: expand in orthonormal basis

$$\langle \vec{u}, \vec{v} \rangle = \left\langle \sum_{i=1}^{N} u_i \hat{e}_i , \sum_{j=1}^{N} v_j \hat{e}_j \right\rangle$$

real vector space $\downarrow$

$$= \sum_{i,j} u_i v_j \langle \hat{e}_i, \hat{e}_j \rangle = \sum_{i,j} u_i v_j \delta_{ij} = \sum_{i=1}^{N} u_i v_i$$

For complex vector space:

$$\langle \vec{u}, \vec{v} \rangle = \sum_{i=1}^{N} u_i^* v_i$$

## Coordinate transformations:

Active transformation: $\vec{v} \to \vec{v}\,' = R\vec{v}$

Write in components:

$$v_i' = \langle \hat{e}_i, \vec{v}\,' \rangle = \langle \hat{e}_i, R\vec{v} \rangle$$

$$= \sum_j \langle \hat{e}_i, R\, v_j \hat{e}_j \rangle = \sum_j \underbrace{\langle \hat{e}_i, R \hat{e}_j \rangle}_{\text{matrix } R_{ij}} v_j$$

$$v_i' = \sum_j R_{ij} v_j$$

$$\begin{pmatrix} v_1' \\ \vdots \\ v_N' \end{pmatrix} = \begin{pmatrix} & R_{ij} & \end{pmatrix} \begin{pmatrix} v_1 \\ \vdots \\ v_N \end{pmatrix}$$

Active transformation transforms coordinates, but leaves basis vectors $\hat{e}_i$ unchanged.

Passive transformation: vector $\vec{v}$ unchanged but basis vectors transformed $\hat{e}_i \to \hat{e}_i' = R\,\hat{e}_i$.

Also changes the components.

$$\vec{v} = \sum_{j=1}^{N} v_j \hat{e}_j = \sum_{j=1}^{N} v_j' \hat{e}_j' = \sum_j v_j' R \hat{e}_j$$

$$\langle \hat{e}_i, \vec{v} \rangle = v_i = \underbrace{\langle \hat{e}_i, R\hat{e}_j \rangle}_{R_{ij}} v_j' = R_{ij} v_j' \Rightarrow v_i' = (R^{-1})_{ij} v_j$$

Active & passive transformations are equivalent.

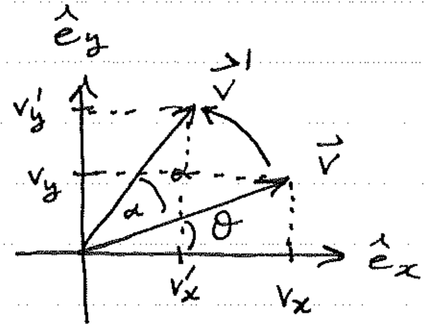
Components transform as inverses in each case.

Here we adopt <u>active</u> point of view: transformation on vector is same as transformation of components. But keep in mind this is equivalent to a change of basis vectors (passive)

Example: Rotations in $\mathbb{R}^2$ (2-dim space)

Vectors $\vec{v} = \begin{pmatrix} v_x \\ v_y \end{pmatrix}$ make up real vector space (dim=2)

Rotate $\vec{v}$ by angle $\alpha$.




active transform.

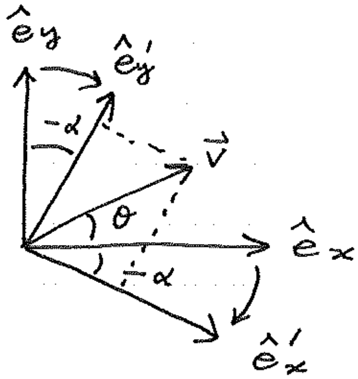
$$\vec{v} = \begin{pmatrix} v_x \\ v_y \end{pmatrix} = |\vec{v}| \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$$

$$\vec{v}' = \begin{pmatrix} v_x' \\ v_y' \end{pmatrix} = |\vec{v}'| \begin{pmatrix} \cos(\theta+\alpha) \\ \sin(\theta+\alpha) \end{pmatrix}$$

Rotation leaves norm invariant: $|\vec{v}| = |\vec{v}'|$

$$\vec{v}' = |\vec{v}| \begin{pmatrix} \cos\theta\cos\alpha - \sin\theta\sin\alpha \\ \sin\theta\cos\alpha + \cos\theta\sin\alpha \end{pmatrix} = |\vec{v}| \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$$

$$= \underbrace{\begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}}_{R(\alpha)\ \text{rotation matrix}} \vec{v}$$




passive transformation

$$\begin{pmatrix} v_x' \\ v_y' \end{pmatrix} = |\vec{v}| \begin{pmatrix} \cos(\theta-\alpha) \\ \sin(\theta-\alpha) \end{pmatrix}$$

$$= \underbrace{\begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix}}_{R^T = R(-\alpha) = R^{-1}(\alpha)} \begin{pmatrix} v_x = |\vec{v}|\cos\theta \\ v_y = |\vec{v}|\sin\theta \end{pmatrix}$$

Rotation matrices are <u>orthogonal matrices</u>: $R^T = R^{-1}$
shown here for $\mathbb{R}^2$ but true for any $\mathbb{R}^n$.

Real vector space: transformations by orthogonal matrices leave inner products invariant.

Transform: $\vec{u} \to \vec{u}' = R\vec{u}$ $\qquad$ $\vec{v} \to \vec{v}' = R\vec{v}$

$u_i \to u_i' = \sum_j R_{ij} u_j$ $\qquad$ $v_i \to v_i' = \sum_j R_{ij} v_j$

$$\langle \vec{u}, \vec{v} \rangle = \sum_i u_i v_i \to \langle \vec{u}', \vec{v}' \rangle = \cancel{\langle R\vec{u}, R\vec{v} \rangle} \sum_i u_i' v_i'$$

$$= \sum_{ijk} R_{ij} u_j R_{ik} v_k$$

$$= \sum_{ijk} u_j (R^T)_{ji} R_{ik} v_k$$

True for any $\vec{u}, \vec{v}$ if $\sum_i (R^T)_{ji} R_{ik} = \delta_{jk}$.

Then in matrix form, we want $R^T R = \mathbb{1}$.
Since $R^{-1}R = \mathbb{1}$, we must have $R^{-1} = R^T$.

Complex vector space: transformations by <u>unitary matrices</u> leave inner product invariant.
Unitary matrix: $R^\dagger = R^{-1}$ where $R^\dagger$ = Hermitian conjugate

## Eigenvalues & eigenvectors

For a square matrix $M$, $\vec{\xi}$ is an eigenvector of $M$ if

$$M\vec{\xi} = \lambda\vec{\xi} \quad (\text{and } \vec{\xi} \neq \vec{0})$$

where $\lambda$ is a scalar. Then $\lambda$ is an eigenvalue of $M$.

Given $M$, need to know how to find eigenvalues & eigenvectors.

Finding eigenvalues $\lambda$: Solve $\det(M - \lambda \mathbb{1}) = 0$.
(characteristic equation)

If $M$ is $N \times N$ matrix, at most $N$ distinct eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_N$.

Finding eigenvectors $\vec{\xi}$: For each eigenvalue $\lambda_a$, solve

$$(M - \lambda_a \mathbb{1})\vec{\xi}_a = 0$$

where $\vec{\xi}_a$ is the corresponding eigenvector.

Each eigenvector has one eigenvalue.
Each eigenvalue has one (or more) eigenvectors.

Note: use $a, b, .. = 1, 2, .. N$ to label different eigenvalues/vectors.
use $i, j, k .. = 1, 2, .. N$ to label components of a vector

e.g. $$\vec{\xi}_a = \begin{pmatrix} \xi_{a1} \\ \xi_{a2} \\ \vdots \\ \xi_{aN} \end{pmatrix} = \sum_{i=1}^{N} \xi_{ai}\,\hat{e}_i$$

Example: $M = \begin{pmatrix} 1 & 2 \\ 2 & -2 \end{pmatrix}$

Characteristic eqn: $0 = \det(M - \lambda \mathbb{1}) = \det\begin{pmatrix} 1-\lambda & 2 \\ 2 & -2-\lambda \end{pmatrix}$

$$= (1-\lambda)(-2-\lambda) - 4$$

$$= \lambda^2 + \lambda - 6 = (\lambda+3)(\lambda-2)$$

Eigenvalues: $\lambda_1 = 2$ and $\lambda_2 = -3$.

Find eigenvectors:

case $\lambda_1 = 2$ : solve $(M - \lambda_1 \mathbb{1})\vec{\xi}_1 = \vec{0}$. Write $\vec{\xi}_1 = \begin{pmatrix} x \\ y \end{pmatrix}$

$$(M - \lambda_1 \mathbb{1})\vec{\xi}_1 = \begin{pmatrix} -1 & 2 \\ 2 & -4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\left.\begin{aligned} -x + 2y &= 0 \\ 2x - 4y &= 0 \end{aligned}\right\} \Rightarrow x = 2y \Rightarrow \vec{\xi}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

Note: eigenvectors are not unique. Can multiply $\vec{\xi}_i$ by any scalar and it is still an eigenvector.

case $\lambda_2 = -3$:

$$(M - \lambda_2 \mathbb{1})\vec{\xi}_2 = \begin{pmatrix} 4 & 2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \left.\begin{aligned} 4x + 2y &= 0 \\ 2x + y &= 0 \end{aligned}\right\} \Rightarrow y = -2x \Rightarrow \vec{\xi}_2 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

Eigenvectors: $\vec{\xi}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$, $\vec{\xi}_2 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$

Example: $M = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix}$

Find eigenvalues:

$$0 = \det(M - \lambda \mathbb{1}) = \det \begin{pmatrix} 1-\lambda & 0 & 1 \\ 0 & 2-\lambda & 0 \\ 1 & 0 & 1-\lambda \end{pmatrix}$$

$$= (1-\lambda)^2(2-\lambda) - (2-\lambda) = (2-\lambda)(1 - 2\lambda + \lambda^2 - 1)$$

$$= (2-\lambda)(\lambda-2)\lambda = -\lambda(\lambda-2)^2$$

Note: only two eigenvalues. $\lambda_1 = 0$ is a normal eigenvalue. But $\lambda_2 = 2$ is a <u>degenerate eigenvalue</u> because it is a repeated root (order 2) in characterstic eqn.

$\lambda_1 = 0$ : single root $\rightarrow$ one eigenvector $\vec{\xi}_1$

$\lambda_2 = 2$ : double root $\rightarrow$ two eigenvectors $\vec{\xi}_2, \vec{\xi}_3$

Case $\lambda_1 = 0$: Write $\vec{\xi}_1 = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$. Solve

$$(M - \lambda_1 \mathbb{1})\vec{\xi}_1 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{array}{l} x + z = 0 \rightarrow x = -z \\ 2y = 0 \rightarrow y = 0 \\ x + z = 0 \end{array}$$

So we have $\vec{\xi}_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$

case $\lambda_2 = 2$: Write $\vec{\xi}_2 = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$.

$$(M - \lambda_2 \mathbb{1})\vec{\xi}_2 = \begin{pmatrix} -1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{array}{l} -x + z = 0 \rightarrow x = z \\ \text{no eqn. for } y \\ (0y = 0, \text{ always true}) \end{array}$$

We have more freedom to choose $\vec{\xi}_2$. Set $y=0$, $x=z=1$.

$$\vec{\xi}_2 = \begin{pmatrix}1\\0\\1\end{pmatrix} \quad \text{is a valid eigenvector.}$$

What about $\vec{\xi}_3$? Want to find a linearly independent $\vec{\xi}_3 = \begin{pmatrix}x\\y\\z\end{pmatrix}$ that satisfies same conditions

$x = z$ & $y =$ anything.

Useful to choose also $\vec{\xi}_3$ such that $\langle \vec{\xi}_2, \vec{\xi}_3 \rangle = 0$.

$$\langle \vec{\xi}_2, \vec{\xi}_3 \rangle = \left\langle \begin{pmatrix}1\\0\\1\end{pmatrix}, \begin{pmatrix}x\\y\\z\end{pmatrix} \right\rangle = x + z = 0 \Rightarrow x = -z$$

since $x = z$ and $x = -z \Rightarrow x = z = 0$.
take $y = 1$.

$$\vec{\xi}_3 = \begin{pmatrix}0\\1\\0\end{pmatrix} \quad \text{works.}$$

So we have
$$\vec{\xi}_1 = \begin{pmatrix}1\\0\\-1\end{pmatrix} \quad \lambda_1 = 0$$
$$\vec{\xi}_2 = \begin{pmatrix}1\\0\\1\end{pmatrix}, \; \vec{\xi}_3 = \begin{pmatrix}0\\1\\0\end{pmatrix} \quad \lambda_2 = 2$$

But note $\vec{\xi}_2$ and $\vec{\xi}_3$ are not unique. Any orthogonal linear combinations are also fine: e.g.

$$\vec{\xi}_2 = \begin{pmatrix}1\\1\\1\end{pmatrix}, \; \vec{\xi}_3 = \begin{pmatrix}1\\-2\\1\end{pmatrix}$$

Hermitian matrices are important in physics

Hermitian matrix : $M = M^\dagger$ or $M_{ij} = M_{ji}^*$

Some facts about Hermitian matrices (assume complex vector space; more general, also includes real vector space as subcase)

<u>Fact 1</u>: Eigenvalues of Hermitian matrices are <u>real</u>.

proof: $M\vec{\xi} = \lambda\vec{\xi} \longrightarrow \vec{\xi}^\dagger M^\dagger = \lambda^* \vec{\xi}^\dagger$ (*)

write: $\langle \vec{\xi}, \vec{\xi} \rangle = \sum_{i=1}^{N} \xi_i^* \xi_i = \vec{\xi}^\dagger \vec{\xi}$

then
$$\begin{aligned} \vec{\xi}^\dagger M \vec{\xi} &= \lambda \vec{\xi}^\dagger \vec{\xi} = \lambda \langle \vec{\xi}, \vec{\xi} \rangle \\ &= \vec{\xi}^\dagger M^\dagger \vec{\xi} \qquad \text{using } (*) \\ &= \lambda^* \vec{\xi}^\dagger \vec{\xi} = \lambda^* \langle \vec{\xi}, \vec{\xi} \rangle \end{aligned}$$

So we have $\lambda \langle \vec{\xi}, \vec{\xi} \rangle = \lambda^* \langle \vec{\xi}, \vec{\xi} \rangle$

Since $\langle \vec{\xi}, \vec{\xi} \rangle > 0$ ($\vec{\xi}$ is not a zero vector), $\lambda = \lambda^*$

<u>Fact 2</u>: Eigenvectors of Hermitian matrices are orthogonal if eigenvalues are distinct.

proof: consider two eigenvectors $\vec{\xi}_1$, $\vec{\xi}_2$ w/ eigenvalues $\lambda_1, \lambda_2$

Evaluate
$$\begin{aligned} \langle \vec{\xi}_1, M\vec{\xi}_2 \rangle = \vec{\xi}_1^\dagger M \vec{\xi}_2 &= \lambda_2 \vec{\xi}_1^\dagger \vec{\xi}_2 = \lambda_2 \langle \vec{\xi}_1, \vec{\xi}_2 \rangle \\ = \vec{\xi}_1^\dagger M^\dagger \vec{\xi}_2 &= \lambda_1 \vec{\xi}_1^\dagger \vec{\xi}_2 = \lambda_1 \langle \vec{\xi}_1, \vec{\xi}_2 \rangle \end{aligned}$$

So $(\lambda_1 - \lambda_2) \langle \vec{\xi}_1, \vec{\xi}_2 \rangle = 0 \Rightarrow \langle \vec{\xi}_1, \vec{\xi}_2 \rangle = 0$ if $\lambda_1 \neq \lambda_2$.

A <u>similarity transformation</u> of a matrix $M$ is defined by $M \to R^{-1} M R$.

<u>Fact 3</u>: A Hermitian matrix $M$ can be <u>diagonalized</u> by a similarity transformation where $R$ is unitary ($R^{-1} = R^{+}$):

$$M \to R^{+} M R = M_d$$

where $M_d = \begin{pmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots \; \lambda_N \end{pmatrix}$ is a diagonal matrix

~~with~~ whose entries are the eigenvalues of $M$.

Proof: Eigenvectors $\vec{\xi}_a$ are orthogonal. Can define unit eigenvectors

$$\hat{\xi}_a = \frac{\vec{\xi}_a}{|\vec{\xi}_a|}, \quad |\vec{\xi}_a| = \sqrt{\langle \vec{\xi}_a, \vec{\xi}_a \rangle}$$

= norm (magnitude) of $\vec{\xi}_a$.

Unit eigenvectors are orthonormal $\langle \hat{\xi}_a, \hat{\xi}_b \rangle = \delta_{ab}$.
Satisfy eigenvalue eqn: $M \hat{\xi}_a = \lambda_a \hat{\xi}_a$.
Have components $\hat{\xi}_{ai}$. Note: $i = 1, \dots, N$ labels components, $a = 1, \dots, N$ labels which eigenvector.

Can treat $\hat{\xi}_{ai}$ as $N \times N$ square matrix: $\hat{\xi}$

Consider $\langle \hat{\xi}_a, M \hat{\xi}_b \rangle = \lambda_b \langle \hat{\xi}_a, \hat{\xi}_b \rangle = \lambda_b \delta_{ab}$

Expanding out in components:

$$\langle \hat{\xi}_a, M \hat{\xi}_b \rangle = \sum_{i,j} \hat{\xi}^*_{ai} M_{ij} \hat{\xi}_{bj} = \lambda_b \delta_{ab}$$

$$= \sum_{i,j} (\hat{\xi}^T)^\dagger_{ai} M_{ji} (\hat{\xi}^T)_{jb} = \lambda_b \delta_{ab}$$

This looks like Matrix multiplication:

$$(\hat{\xi}^T)^\dagger M (\hat{\xi}^T) = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_N \end{pmatrix} = M_d$$

The unitary matrix that diagonalizes $M$ is $R = \hat{\xi}^T$.
i.e. $R$ is a matrix whose components are $R_{ia} = \hat{\xi}_{ai}$.

$$R = \underbrace{\left( \begin{pmatrix} \\ \hat{\xi}_1 \\ \\ \end{pmatrix} \begin{pmatrix} \\ \hat{\xi}_2 \\ \\ \end{pmatrix} \cdots \begin{pmatrix} \\ \hat{\xi}_N \\ \\ \end{pmatrix} \right)}_{N \times N}$$

example: $M = \begin{pmatrix} 1 & 2 \\ 2 & -2 \end{pmatrix}$. We found previously

$$\vec{\xi}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \ \& \ \lambda_1 = 2 \ ; \ \vec{\xi}_2 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}, \ \lambda_2 = -3.$$

$$\Rightarrow \hat{\xi}_1 = \frac{1}{\sqrt{5}} \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \quad \hat{\xi}_2 = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

$$R = \left( \begin{pmatrix} \hat{\xi}_1 \end{pmatrix} \begin{pmatrix} \hat{\xi}_2 \end{pmatrix} \right) = \frac{1}{\sqrt{5}} \begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix} \quad ✓$$

$$R^\dagger M R = \frac{1}{5} \begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 2 & -2 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix} = \frac{1}{5} \begin{pmatrix} 4 & 2 \\ -3 & +6 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & -3 \end{pmatrix}$$

$$\cancel{= \frac{1}{5}}$$

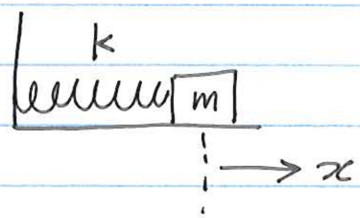
Special case: real vector space. All matrices & vectors are real.

| | complex | real |
|---|---|---|
| • | M is Hermitian <br> $M = M^{\dagger} = (M^T)^*$ | M is real symmetric <br> $M = M^*$ (real) <br> $M = M^T$ (sym.) |
| • | Eigenvalues $\lambda$ = real | Eigenvalues $\lambda$ = real |
| • | M diagonalized by unitary matrix $R$ <br> $(R^{-1} = R^{\dagger})$ | M diagonalized by ~~real symmetric~~ orthogonal matrix $R$ <br> $(R^{-1} = R^T)$ |

## Coupled systems

Eigenvalue methods very useful for solving coupled differential equations.

First consider simple harmonic oscillator




$x$ = one-dim. displacement from equilibrium

Potential energy $V = \frac{1}{2}kx^2$

Equation of motion (Newton's 2nd law)

$$m\ddot{x} = -\frac{\partial V}{\partial x} = -kx \qquad \left(\ddot{x} = \frac{d^2x}{dt^2}\right)$$

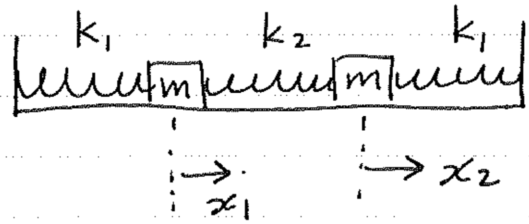
$$\ddot{x} = -\omega^2 x$$

where $\omega = \sqrt{k/m}$ is oscillation frequency.

Solution: $x(t) = A\sin(\omega t) + B\cos(\omega t)$

$A, B$ determined by initial conditions.

Coupled harmonic oscillator:




$x_1, x_2$ = displacements from equilibrium

$k_1, k_2$ = spring constants

equal masses $m$.

Potential energy:

$$V = \underbrace{\tfrac{1}{2} k_1 x_1^2}_{\text{spring}} + \underbrace{\tfrac{1}{2} k_2 (x_1 - x_2)^2}_{\text{Spring 2}} + \underbrace{\tfrac{1}{2} k_1 x_2^2}_{\text{spring 3}}$$

Equation of motion:

$$m \ddot{x}_1 = \underbrace{-\frac{\partial V}{\partial x_1}}_{\text{Force on mass 1}} = -\left(k_1 x_1 + k_2 (x_1 - x_2)\right)$$

$$m \ddot{x}_2 = \underbrace{-\frac{\partial V}{\partial x_2}}_{\text{Force on mass 2}} = -\left(+k_2 (x_2 - x_1) + k_1 x_2\right)$$

Write as matrix equation:

$$\begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} = -\begin{pmatrix} \frac{k_1+k_2}{m} x_1 - \frac{k_2}{m} x_2 \\ -\frac{k_2}{m} x_1 + \frac{k_1+k_2}{m} x_2 \end{pmatrix}$$

$$= -\underbrace{\begin{pmatrix} \frac{k_1+k_2}{m} & -\frac{k_2}{m} \\ -\frac{k_2}{m} & \frac{k_1+k_2}{m} \end{pmatrix}}_{U} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Also define vector $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

So we have eqn. of motion $\ddot{\vec{x}} = -U\vec{x}$

Want to solve for $x_1(t)$ & $x_2(t)$. Diagonalize equations of motion.

Step 1: Find eigenvalues. Write $U = \begin{pmatrix} a & -b \\ -b & a \end{pmatrix}$ where $a = \frac{k_1+k_2}{m}$ and $b = \frac{k_2}{m}$.

Eigenvalues are:

$$0 = \det(U - \lambda \mathbb{1}) = \det\begin{pmatrix} a-\lambda & -b \\ -b & a-\lambda \end{pmatrix} = (a-\lambda)^2 - b^2$$

$$\Rightarrow \quad \lambda^2 - 2a\lambda + a^2 - b^2 = 0$$

$$\lambda = \frac{1}{2}\left[2a \pm \sqrt{4a^2 - 4(a^2-b^2)}\right] = a \pm b$$

$$\lambda = \begin{cases} a - b = k_1/m = \omega_1^2 \\ a + b = \frac{k_1+2k_2}{m} = \omega_2^2 \end{cases}$$

Step 2: Find eigenvectors.

case: $\lambda_1 = \omega_1^2 = a - b$. Write $\vec{\xi}_1 = \begin{pmatrix} x \\ y \end{pmatrix}$

$$(U - \lambda_1 \mathbb{1})\vec{\xi}_1 = \begin{pmatrix} b & -b \\ -b & b \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{matrix} bx - by = 0 \\ x = y \end{matrix}$$

$$\vec{\xi}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

case : $\lambda_2 = \omega_2^2 = a+b$

$$(U - \lambda_2 \mathbb{1})\vec{\xi}_2 = \begin{pmatrix} -b & -b \\ -b & -b \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{matrix} -bx-by=0 \\ x=-y \end{matrix}$$

So we have $\vec{\xi}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

Step 3: Write down solution

First introduce <u>normal modes</u> :

$$\vec{x}_a(t) = \vec{\xi}_a \left( A \sin(\omega_a t) + B \cos(\omega_a t) \right)$$

$$\vec{x}_1(t) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \left( A \sin(\omega_1 t) + B \cos(\omega_1 t) \right)$$

$$= \begin{pmatrix} A \sin(\omega_1 t) + B \cos(\omega_1 t) \\ A \sin(\omega_1 t) + B \cos(\omega_1 t) \end{pmatrix} = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$$

$$\vec{x}_2(t) = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \left( A \sin(\omega_2 t) + B \cos(\omega_2 t) \right)$$

$$= \begin{pmatrix} A \sin(\omega_2 t) + B \cos(\omega_2 t) \\ -A \sin(\omega_2 t) - B \cos(\omega_2 t) \end{pmatrix} = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$$

Normal modes are solutions to Eqn. of motion

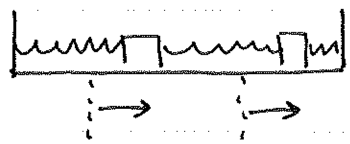
$$\ddot{\vec{x}}_a = -\omega_a^2 \vec{\xi}_a \left( A \sin(\omega_a t) + B \cos(\omega_a t) \right) = -\omega_a^2 \vec{x}_a$$

$$= -U \vec{x}_a = -U \vec{\xi}_a \left( A \sin(\omega_a t) + B \cos(\omega_a t) \right)$$

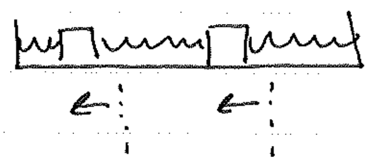
$$= -\omega_a^2 \vec{\xi}_a \left( A \sin(\omega_a t) + B \cos(\omega_a t) \right) = -\omega_a^2 \vec{x}_a$$

What is the meaning of normal modes?

$\vec{x}_1(t)$:

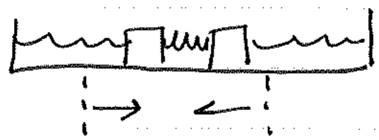


$x_1 = x_2$

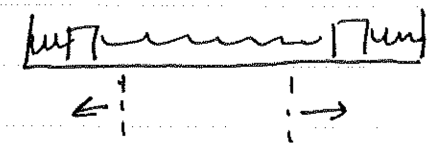


$x_1 = x_2$

$\vec{x}_2(t)$:



$x_1 = -x_2$



$x_1 = -x_2$

Both types of oscillations each oscillate with single frequency & don't mix with one another

General solution: linear combination of normal modes.*

$$\vec{x}(t) = \vec{\xi}_1 \left(A_1 \sin(\omega_1 t) + B_1 \cos(\omega_1 t)\right)$$
$$+ \vec{\xi}_2 \left(A_2 \sin(\omega_2 t) + B_2 \cos(\omega_2 t)\right)$$

Coefficients $A_1, B_1, A_2, B_2$ fixed by initial conditions.

Why does this work? Another perspective:

Original eqn: $\ddot{\vec{x}} = -U\vec{x}$

Diagonalize $U$ with (unitary) ~~eexr~~ similarity transformation:

$$R^\dagger U R = \begin{pmatrix} \omega_1^2 & 0 \\ 0 & \omega_2^2 \end{pmatrix} = U_d \Rightarrow R U_d R^\dagger = U$$

Define transformed vector $\vec{x}' = R^{\dagger}\vec{x}$ or $\vec{x} = R\vec{x}'$

Then $\ddot{\vec{x}}' = R^{\dagger}\ddot{\vec{x}} = -R^{\dagger}U\vec{x} = -R^{\dagger}UR\vec{x}' = -U_d\vec{x}'$

In terms of $\vec{x}' = \begin{pmatrix} x_1' \\ x_2' \end{pmatrix}$, the equation of motion is two <u>uncoupled</u> equations

$$\ddot{\vec{x}}' = \begin{pmatrix} \ddot{x}_1' \\ \ddot{x}_2' \end{pmatrix} = -\begin{pmatrix} \omega_1^2 & 0 \\ 0 & \omega_2^2 \end{pmatrix}\begin{pmatrix} x_1' \\ x_2' \end{pmatrix}$$

$$\underbrace{\ddot{x}_1' = -\omega_1^2 x_1' \qquad \ddot{x}_2' = -\omega_2^2 x_2'}_{\text{two simple harmonic oscillators.}}$$

$$x_1'(t) = A_1 \sin(\omega_1 t) + B_1 \cos(\omega_2 t)$$
$$x_2'(t) = A_2 \sin(\omega_2 t) + B_2 \cos(\omega_2 t)$$

Now rotate back to $\vec{x} = R\vec{x}'$

$$\vec{x} = R\begin{pmatrix} A_1 \sin(\omega_1 t) + B_1 \cos(\omega_2 t) \\ A_2 \sin(\omega_2 t) + B_2 \cos(\omega_2 t) \end{pmatrix}, \quad \text{write } R = \begin{pmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{pmatrix}$$

$$= \underbrace{\begin{pmatrix} R_{11} \\ R_{21} \end{pmatrix}}_{\hat{\xi}_1}\left(A_1 \sin(\omega_1 t) + B_1 \cos(\omega_2 t)\right) + \underbrace{\begin{pmatrix} R_{12} \\ R_{22} \end{pmatrix}}_{\hat{\xi}_2}\left(A_2 \sin(\omega_2 t) + B_2 \cos(\omega_2 t)\right)$$

Not all systems have sine & cosine solutions.

Diagonalized eqn of motion

$\ddot{x}_a' = -\omega_a^2 x_a'$ $(a=1,2)$ $\rightarrow$ sine/cosine for $x_a'(t)$

other eqn of motion $\rightarrow$ other solution

Example: radioactive decay $\quad 1 \xrightarrow{\lambda_1} 2 \xrightarrow{\lambda_2} \ldots$

$$\dot{n}_1 = -\lambda_1 n_1$$
$$\dot{n}_2 = \lambda_1 n_1 - \lambda_2 n_2$$

Define $\vec{n} = \begin{pmatrix} n_1 \\ n_2 \end{pmatrix}$. Write as matrix equation

$$\dot{\vec{n}} = -\underbrace{\begin{pmatrix} \lambda_1 & 0 \\ -\lambda_1 & \lambda_2 \end{pmatrix}}_{M} \vec{n} = -M\vec{n}$$

Note: M is not Hermitian. Still can use eigenvalue methods.

Find eigenvalues: $\quad 0 = \det(M - \mathbb{1}\lambda) = \det\begin{pmatrix} \lambda_1-\lambda & 0 \\ -\lambda_1 & \lambda_2-\lambda \end{pmatrix}$
$$= (\lambda_1-\lambda)(\lambda_2-\lambda)$$

Eigenvalues are simply $\lambda = \lambda_1, \lambda_2$.

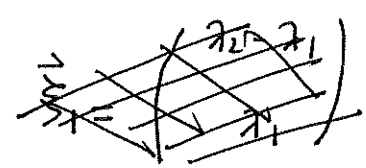
Find eigenvectors:

$\lambda = \lambda_1$: $\quad M\vec{\xi}_1 = \lambda_1 \vec{\xi}_1$. Write $\vec{\xi}_1 = \begin{pmatrix} x \\ y \end{pmatrix}$.

$$(M - \lambda_1 \mathbb{1})\vec{\xi}_1 = \begin{pmatrix} 0 & 0 \\ -\lambda_1 & \lambda_2-\lambda_1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$-\lambda_1 x + (\lambda_2 - \lambda_1)y = 0$$

$$\lambda_1 x = (\lambda_2 - \lambda_1)y \Rightarrow$$



$$\vec{\xi}_1 = \begin{pmatrix} 1 \\ \dfrac{\lambda_2-\lambda_1}{\lambda_1} \end{pmatrix}$$

$$\lambda = \lambda_2 : \quad (M - \lambda_2 \mathbb{1})\vec{\xi}_2 = \begin{pmatrix} \lambda_1 - \lambda_2 & 0 \\ -\lambda_1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow (\lambda_1 - \lambda_2)x = 0 \ \& \ -\lambda_1 x = 0 \Rightarrow \begin{array}{l} x = 0 \\ y = \text{anything} \end{array}$$

Take $\vec{\xi}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

Normal modes are:

$$\vec{n}_1(t) = \vec{\xi}_1 \cdot A_1 e^{-\lambda_1 t}, \quad \vec{n}_2(t) = \vec{\xi}_2 A_2 e^{-\lambda_2 t}$$

Check eqn of motion:

$$\text{LHS} = \dot{\vec{n}}_a(t) = -\lambda_a \vec{\xi}_a A_a e^{-\lambda_a t} = -\lambda_a \vec{n}_a(t)$$

$$\text{RHS} = -M\vec{n}_a(t) = -M\vec{\xi}_a A_a e^{-\lambda_a t} = -\lambda_a \vec{n}_a(t)$$

Works!

General solution:

$$\vec{n}(t) = \vec{\xi}_1 A_1 e^{-\lambda_1 t} + \vec{\xi}_2 A_2 e^{-\lambda_2 t}$$

Initial condition: $n_1(0) = N$, $n_2(0) = 0$. $\Rightarrow \vec{n}(0) = \begin{pmatrix} N \\ 0 \end{pmatrix}$

$$\vec{n}(0) = \begin{pmatrix} 1 \\ \frac{\lambda_2 - \lambda_1}{\lambda_1} \end{pmatrix} A_1 e^{-\lambda_1 t \to 0} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} A_2 e^{-\lambda_2 t \to 0}$$

$$= \begin{pmatrix} A_1 \\ A_2 + \frac{\lambda_2 - \lambda_1}{\lambda_1} A_1 \end{pmatrix} = \begin{pmatrix} N \\ 0 \end{pmatrix}$$

$$A_1 = N, \qquad A_2 = \frac{\lambda_1 - \lambda_2}{\lambda_1} N$$

$$\vec{n}(t) = N \vec{\xi}_1 e^{-\lambda_1 t} + N \frac{\lambda_1 - \lambda_2}{\lambda_1} \vec{\xi}_2 e^{-\lambda_2 t}$$

$$= \begin{pmatrix} N e^{-\lambda_1 t} \\ N \left( \frac{\lambda_1 - \lambda_2}{\lambda_1} \right) \left( e^{-\lambda_2 t} - e^{-\lambda_1 t} \right) \end{pmatrix} = \begin{pmatrix} n_1(t) \\ n_2(t) \end{pmatrix}$$

What about diagonalization? Can still diagonalize using similarity transform:

Define $\vec{n}' = R^{-1} \vec{n}$ and $\vec{n} = R \vec{n}'$

$$R^{-1} M R = M_d = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

$$\dot{\vec{n}}' = R^{-1} \dot{\vec{n}} = -R^{-1} M \vec{n} = -R^{-1} M R \vec{n}' = -M_d \vec{n}'$$

Still diagonalized eqn of motion using $R$:

$$\dot{\vec{n}}' = \begin{pmatrix} \dot{n}_1' \\ \dot{n}_2' \end{pmatrix} = -\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} n_1' \\ n_2' \end{pmatrix}$$

Solutions (2 uncoupled first order eqns):

$$n_1'(t) = A_1 e^{-\lambda_1 t}$$

$$n_2'(t) = A_2 e^{-\lambda_2 t}$$

Then return to original basis:

$$\vec{n} = R\,\vec{n}' = \begin{pmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{pmatrix} \begin{pmatrix} n_1'(t) \\ n_2'(t) \end{pmatrix}$$

$$= \begin{pmatrix} R_{11}\,n_1' + R_{12}\,n_2' \\ R_{21}\,n_1' + R_{22}\,n_2' \end{pmatrix} = \begin{pmatrix} R_{11} \\ R_{21} \end{pmatrix} n_1'(t) + \begin{pmatrix} R_{12} \\ R_{22} \end{pmatrix} n_2'(t)$$

$$= \underbrace{\begin{pmatrix} R_{11} \\ R_{21} \end{pmatrix}}_{\vec{\xi}_1} A_1\, e^{-\lambda_1 t} + \underbrace{\begin{pmatrix} R_{12} \\ R_{22} \end{pmatrix}}_{\vec{\xi}_2} A_2\, e^{-\lambda_2 t}$$

Rotation matrix is given by $R = \left( (\vec{\xi}_1)(\vec{\xi}_2) \right) = \begin{pmatrix} 1 & 0 \\ \frac{\lambda_2 - \lambda_1}{\lambda_1} & 1 \end{pmatrix}$

Differences: $R$ is no longer unitary since $M$ not Hermitian
$\langle \vec{\xi}_1, \vec{\xi}_2 \rangle \neq 0$, eigenvectors not orthogonal.
Don't need to normalize eigenvectors. This is only required if we want a unitary $R$, which is not possible here.