PHYS 5180: Homework 2 (due Friday 4pm Jan. 30)

1. Consider an N-particle state

$$|\mathbf{p}_1, \mathbf{p}_2, ..., \mathbf{p}_N\rangle = \left(\prod_{i=1}^{N} \sqrt{2E_i} a^\dagger_{\mathbf{p}_i}\right) |0\rangle \,. \tag{1}$$

Show that

$$\begin{aligned} H|\mathbf{p}_1, \mathbf{p}_2, ..., \mathbf{p}_N\rangle &= \left(\sum_{i=1}^{N} E_i\right) |\mathbf{p}_1, \mathbf{p}_2, ..., \mathbf{p}_N\rangle \\ \mathbf{P}|\mathbf{p}_1, \mathbf{p}_2, ..., \mathbf{p}_N\rangle &= \left(\sum_{i=1}^{N} \mathbf{p}_i\right) |\mathbf{p}_1, \mathbf{p}_2, ..., \mathbf{p}_N\rangle \end{aligned}$$

where H is the Hamiltonian and $\mathbf{P}$ is the momentum operator.

2. A Lorentz transformation Λ can be represented by a transformation operator $U(\Lambda)$ acting on a state $|\mathbf{p}\rangle$ as follows:

$$U(\Lambda)|\mathbf{p}\rangle = |\mathbf{p}'\rangle = |\Lambda\mathbf{p}\rangle$$

where $\mathbf{p}' = \Lambda\mathbf{p}$ is shorthand for $p'^\mu = \Lambda^\mu{}_\nu p^\nu$. Show the following:

- $U(\Lambda)$ must be a unitary operator.
- The creation and annihilation operators transform as

$$\begin{aligned} a_{\mathbf{p}'} &= U(\Lambda) a_{\mathbf{p}} U(\Lambda)^\dagger \sqrt{E_{\mathbf{p}}/E_{\mathbf{p}'}} \\ a^\dagger_{\mathbf{p}'} &= U(\Lambda) a^\dagger_{\mathbf{p}} U(\Lambda)^\dagger \sqrt{E_{\mathbf{p}}/E_{\mathbf{p}'}} \end{aligned}$$

- $U(\Lambda)\phi(x)U(\Lambda)^\dagger = \phi(\Lambda x)$.

3. Consider the Lagrangian for a complex scalar field with a classical external source $j(x)$:

$$\mathscr{L}(x) = |\partial_\mu \phi(x)|^2 - m^2|\phi(x)|^2 + j(x)\phi(x)^* + j(x)^*\phi(x)\,. \tag{2}$$

Suppose the source term is zero in the distant past, turns on for some finite amount of time, and then turns off again. Assuming we start in the vacuum state $|0\rangle$ before the source turns on, compute the total energy and charge, given by

$$\langle 0|H|0\rangle\,, \qquad \langle 0|Q|0\rangle\,,$$

after the source has turned off. *Hint:* See PS pages 32-33 for the case of a real scalar field.