PHYS 5180: Homework 7 (due Friday 4pm Apr. 3)

1. PS problem 6.1.

2. This problem will illustrate the **Higgs mechanism** in a simpler context.

- Consider the Lagrangian for a vector field A_μ with a mass term:

  $$\mathscr{L} = -\tfrac{1}{4} F_{\mu\nu} F^{\mu\nu} + m_A^2 A_\mu A^\mu \,.$$

  Show that the equation of motion (in Lorentz gauge) is $(\partial^2 + m_A^2) A^\mu = 0$. Show, however, that $\mathscr{L}$ is not gauge invariant unless $m_A = 0$.

- The point of the Higgs mechanism is to generate $m_A \neq 0$ from a gauge-invariant theory. Consider the Lagrangian for a *massless* vector field A^μ and a complex scalar ϕ with charge $q_\phi e$:

  $$\mathscr{L} = |D_\mu \phi|^2 - V(\phi) - \tfrac{1}{4} F_{\mu\nu} F^{\mu\nu} \,,$$

  where the covariant derivative is $D_\mu = \partial_\mu - i q_\phi e A_\mu$, and the potential is

  $$V(\phi) = -\frac{\mu^2}{2} \phi^\dagger \phi + \frac{\lambda}{4} (\phi^\dagger \phi)^2$$

  where $\mu^2, \lambda > 0$. Show that $\mathscr{L}$ is invariant under local (gauge) transformations

  $$\phi(x) \to e^{-i q_\phi \alpha(x)} \phi(x) \,, \qquad A_\mu(x) \to A_\mu(x) - \frac{1}{e} \partial_\mu \alpha(x) \,.$$

  Show that the potential $V(\phi)$ is minimized for $|\phi| = v/\sqrt{2} \neq 0$ and determine v in terms of μ^2 and λ.[1]

- Since the vacuum corresponds to the minimum energy state, we must expand the theory about the point $|\phi| = v/\sqrt{2}$. Write

  $$\phi = \Big(\frac{v + h(x)}{\sqrt{2}} \Big) e^{i\eta(x)/v} \qquad (1)$$

  where $h(x)$ and $\eta(x)$ are two real scalar fields, corresponding to fluctuations about the vacuum. Express $\mathscr{L}$ in terms of these fields, and evaluate the masses of h and η. Next, show that η may be removed from the theory by performing a gauge transformation, and that $\mathscr{L}$ now describes the theory of a *massive* vector boson A^μ and one *real* scalar field h.[2] Determine m_A.

[1] v is known as the "vacuum expectation value" (vev) since $\langle \phi \rangle = v$.

[2] In the original theory, the complex scalar ϕ and massless vector A^μ each correspond to two degrees of freedom. In the new theory, the massive A^μ has three degrees of freedom (two transverse + one longitudinal polarization) and the real scalar has one degree of freedom. We say that A^μ has **eaten** the massless field η to acquire its mass.

- Evaluate the partial decay rate for $h \to AA$ in terms of m_h, m_A, and v. Note: it is necessary to use a modified polarization sum rule for a massive vector boson, which has three physical polarizations:

$$\sum_{\text{pol}} \varepsilon_\mu(k)\varepsilon^*_\nu(k) \to -\left(g_{\mu\nu} - \frac{k_\mu k_\nu}{m_A^2}\right).$$

  This rule can be verified in the A rest frame using the explicit polarization vectors $\epsilon = (0,1,0,0)$, $(0,0,1,0)$, $(0,0,0,1)$, while the unphysical time-like polarization $\epsilon = (1,0,0,0)$ does not contribute.

- Next, we consider fermions. Suppose that the gauge interaction is **chiral**, meaning that left- and right-handed chiral fields transform differently:

$$\psi_L(x) \to e^{-iq_L\alpha(x)}\psi_L(x), \qquad \psi_R(x) \to e^{-iq_R\alpha(x)}\psi_R(x),$$

  where $q_{L,R}$ are the chiral charges. Show that the standard Dirac mass term

$$m\bar{\psi}\psi$$

  is forbidden by the gauge symmetry, but a Yukawa interaction is allowed

$$\mathscr{L}_{\text{int}} = -g_\psi \bar{\psi}_R \psi_L \phi + \text{h.c.}, \tag{2}$$

  assuming $q_\phi + q_L = q_R$, where g_ψ is a real, positive coupling constant. Show that by making the substition in Eq. (1), the Yukawa interaction (2) generates a mass term for ψ.

- Determine the Feynman rules for the $h\bar{f}f$ and $A\bar{f}f$ vertices and compute the decay rates $\Gamma(h \to f\bar{f})$ and $\Gamma(A \to f\bar{f})$, assuming these decays are kinematically allowed, in terms of m_f, m_A, m_h, and v.

This problem is very similar to the actual Standard Model (which has a more complicated gauge symmetry), with A being a proxy for the W, Z bosons. This shows how the Higgs field generates masses for gauge bosons and fermions, which are otherwise forbidden by gauge symmetry.

3. One hypothesis for dark matter is that it may interact through the Higgs boson. In this case, dark matter passing through the Earth may be observable by scattering with nuclei in detectors (known as **direct detection experiments**).

Consider the following interaction (valid at low energy)

$$\mathscr{L}_{\text{int}} = -g_n \bar{n} n h - g_\chi \bar{\chi}\chi h \tag{3}$$

where n is a Dirac fermion for the nucleon (proton or neutron), χ is a Dirac fermion for dark matter, and h is the Higgs boson (a real scalar). The nucleon-Higgs coupling is known to be $g_n \approx 10^{-3}$, but the dark matter mass m_χ and coupling g_χ are unknown.

- Show that the unpolarized cross section (known as the **spin-independent** cross section) for $\chi n \to \chi n$ in the nonrelativisitic limit is

$$\sigma(n\chi \to n\chi) = \frac{g_\chi^2 g_n^2 \mu_{n\chi}^2}{\pi m_h^4}, \tag{4}$$

  where $\mu_{n\chi}$ is the dark matter-nucleon reduced mass.

- Next, we will consider the scattering cross section for $N\chi \to N\chi$, where N is a nucleus with A nucleons. Assuming all particles are nonrelativistic and the initial N is at rest, show that the momentum transfer is $|\mathbf{q}| < 2\mu_{N\chi} v_{\rm dm}$, where $v_{\rm dm} \sim 10^{-3}$ c is the initial dark matter velocity. For $m_\chi \sim 100$ GeV and $A \sim 100$, argue that the de Broglie wavelength $\lambda = 2\pi/|\mathbf{q}|$ is comparable to the size of the nucleus. This implies that dark matter scatters **coherently** from the entire nucleus, not individual nucleons. Therefore, compute $\sigma(N\chi \to N\chi)$ by replacing

$$g_n \to g_N = A g_n , \qquad \mu_{n\chi} \to \mu_{N\chi} \, . \tag{5}$$

  in your result above.

- For $m_\chi = 100$ GeV, compute the expected number of dark matter-nucleus collisions as a function of g_χ given the following information: the mass density of dark matter particles passing through the Earth is around $\rho_{\rm dm} \approx 0.3$ GeV/cm^3, the detector is 100 kg with $A = 100$,[3] and the experiment will operate for 100 days. Assuming no background and no signal events (i.e. less than 1 event) were observed, what is approximate limit on g_χ?

- For lighter mass dark matter, direct detection experiments have reduced sensitivity due to dark matter-nuclear scattering imparting too little energy to be detected. However, Higgs bosons produced at the Large Hadron Collider can study this case (provided $m_\chi < m_h/2$). Compute the partial width for $\Gamma(h \to \chi\bar{\chi})$, known as the **invisible Higgs width** since $\chi\bar{\chi}$ are unobserved in the collider experiment. Given the current (approximate) experimental constraint

$$\Gamma(h \to \text{invisible states}) \lesssim 2 \text{ MeV} \, , \tag{6}$$

  determine the constraint on g_χ assuming $m_\chi \ll m_h$.

[3] Typically materials are xenon, argon, or germanium.