Quantum Field theory

I. Dimensional units

"particle physics units" $\hbar = c = 1$.

All dimensional parameters have units of energy (eV)

$$1 \text{ eV} \cong 1.6 \times 10^{-19} \text{ J} \quad (\text{J} = \text{kg m}^2/\text{s}^2)$$
$$1 \text{ keV} = 10^3 \text{ eV}$$
$$1 \text{ MeV} = 10^6 \text{ eV}$$
$$1 \text{ GeV} = 10^9 \text{ eV}$$

Mass and momentum have units of energy (mass dimension 1)

Length and time have units of 1/energy (mass dimension −1)

Dimensional analysis:

- electron mass $$m_e = 9.11 \times 10^{-31} \text{ kg} \times \left(\frac{1 \text{ eV}}{1.6 \times 10^{-19} \text{ kg m}^2/\text{s}^2} \right) \underbrace{\left(\frac{3 \times 10^8 \text{ m}}{\text{s}} \right)^2}_{c^2}$$
$$= 511 \text{ keV}$$

- $1 \text{ fm} = 10^{-15} \text{ m}$ (approx proton radius)
$$= 10^{-15} \text{ m} \times \underbrace{\frac{1}{1.05 \times 10^{-34} \text{ J s}}}_{\hbar} \; \underbrace{\frac{1}{3 \times 10^8 \text{ m/s}}}_{c} \times \left(\frac{1.6 \times 10^{-19} \text{ J}}{1 \text{ eV}} \right)$$
$$= 0.005 \text{ MeV}^{-1} = \frac{1}{197 \text{ MeV}}$$

useful conversion factor $1 = \hbar c \cong 197 \text{ MeV fm}$

II. Special relativity and Lorentz transformations

Four-vectors:

4-momentum $p^\mu = (E, \underline{p})$ $\quad E =$ energy, $\underline{p} =$ 3-momentum

spacetime coordinate $x^\mu = (t, \underline{x})$ $\quad t =$ time, $\underline{x} =$ position.

Greek indices $\mu, \nu, \lambda, \ldots = 0, 1, 2, 3$ (Lorentz indices)

$$p^0 = E, \quad x^0 = t$$

$$\underline{p} = (p^1, p^2, p^3)$$

Roman indices $i, j, k, \ldots = 1, 2, 3$ (spatial indices)

$$\left.\begin{array}{l} p^i = \underline{p} \\ x^i = \underline{x} \end{array}\right\} \text{ 3-vectors}$$

Scalar product:

$$p \cdot q = p^0 q^0 - p^1 q^1 - p^2 q^2 - p^3 q^3$$

$$= (p^0, \underline{p}) \underbrace{\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}}_{\text{metric tensor } g_{\mu\nu} = \begin{pmatrix} +1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}} \begin{pmatrix} q^0 \\ \underline{q} \end{pmatrix}$$

$$= \sum_{\mu,\nu=0}^{3} g_{\mu\nu} p^\mu q^\nu = g_{\mu\nu} p^\mu q^\nu \quad \text{(Einstein summation convention)}$$

A repeated index is summed over (one raised, one lowered)

Four vectors with Lowered indices:

$$p_\mu = g_{\mu\nu}\, p^\nu = (E, -\underline{p})$$
$$x_\mu = g_{\mu\nu}\, x^\nu = (t, -\underline{x})$$

Lowering (or raising) index flips sign of spatial component

$$p^\mu = g^{\mu\nu} p_\nu \qquad \text{so } g^{\mu\nu} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix} = g_{\mu\nu}$$

Note:

$$g^\mu{}_\nu = g^{\mu\lambda} g_{\lambda\nu} = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} = \delta^\mu_\nu \quad \text{Kronecker delta}$$

also $g_\nu{}^\mu = \delta^\mu_\nu$

Derivatives:

$$\left.\begin{aligned} \partial_\mu &\equiv \frac{\partial}{\partial x^\mu} = \left(\frac{\partial}{\partial t}, \underline{\nabla}\right) \\ \partial^\mu &= \frac{\partial}{\partial x_\mu} = \left(\frac{\partial}{\partial t}, -\underline{\nabla}\right) \end{aligned}\right\} \text{ opposite to a normal 4-vector}$$

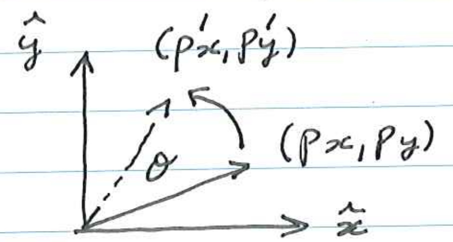
Required by $\partial_\mu x^\nu = \dfrac{\partial x^\nu}{\partial x^\mu} = \delta^\nu_\mu = g_\mu{}^\nu$

Similarly: $\partial_\mu x_\nu = g_{\mu\nu}$

$$\partial_\mu (p \cdot x) = \partial_\mu (p^\nu x_\nu) = g_{\mu\nu}\, p^\nu = p_\mu$$

Lorentz transformations: group of rotations & boost acting on 4-vectors.

- rotations: e.g. rotate about $\hat{z}$ axis by angle θ



$$p_x' = p_x \cos\theta - p_y \sin\theta$$
$$p_y' = p_y \cos\theta + p_x \sin\theta$$

In 4-vector notation:

$$p^\mu \to p'^\mu = \begin{pmatrix} E \\ p^1\cos\theta - p^2\sin\theta \\ p^2\cos\theta + p^1\sin\theta \\ p^3 \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & c\theta & -s\theta & 0 \\ 0 & s\theta & c\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}}_{\Lambda^\mu{}_\nu} \underbrace{\begin{pmatrix} E \\ p^1 \\ p^2 \\ p^3 \end{pmatrix}}_{p^\nu}$$

$$p^\mu \to p'^\mu = \Lambda^\mu{}_\nu \, p^\nu \quad \text{(Lorentz transform)}$$

Scalar product: $p\cdot q \to p'\cdot q' = g_{\mu\nu} p'^\mu q'^\nu$

$$= g_{\mu\nu} \Lambda^\mu{}_\lambda \Lambda^\nu{}_\kappa \, p^\lambda q^\kappa$$

Evaluate $g_{\mu\nu} \Lambda^\mu{}_\lambda \Lambda^\nu{}_\kappa$ by matrix multiplication.

$$g_{\mu\nu} \Lambda^\mu{}_\lambda \Lambda^\nu{}_\kappa = \underbrace{\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & c\theta & s\theta & 0 \\ 0 & -s\theta & c\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}}_{(\Lambda^\mu{}_\lambda)^T} \underbrace{\begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}}_{g_{\mu\nu}} \underbrace{\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & c\theta & -s\theta & 0 \\ 0 & s\theta & c\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}}_{\Lambda^\nu{}_\kappa}$$

$$= \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix} = g_{\lambda\kappa}$$

So $p\cdot q \to p'\cdot q' = g_{\lambda\kappa}\, p^\lambda q^\kappa = p\cdot q$

Scalar product unchanged by rotations.

- boosts: e.g. boost along $\hat{z}$ axis by β

$$E \to \gamma(E + \beta p_z)$$
$$p_z \to \gamma(p_z + \beta E)$$
p_x, p_y unchanged

In 4-vector notation:

$$p^\mu \to p'^\mu = \begin{pmatrix} \gamma E + \beta p^3 \\ p^1 \\ p^2 \\ \gamma\beta E + \gamma p^3 \end{pmatrix} = \underbrace{\begin{pmatrix} \gamma & 0 & 0 & \beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \beta\gamma & 0 & 0 & \gamma \end{pmatrix}}_{\Lambda^\mu{}_\nu} \underbrace{\begin{pmatrix} E \\ p^1 \\ p^2 \\ p^3 \end{pmatrix}}_{p^\nu}$$

So again $p^\mu \to p'^\mu = \Lambda^\mu{}_\nu\, p^\nu$

Scalar product: $p\cdot q \to p'\cdot q' = g_{\mu\nu}\, \Lambda^\mu{}_\lambda\, \Lambda^\nu{}_\kappa\, p^\lambda q^\kappa$

Again compute $g_{\mu\nu}\, \Lambda^\mu{}_\lambda\, \Lambda^\nu{}_\kappa$ by matrix multiplication

$$g_{\mu\nu}\, \Lambda^\mu{}_\lambda\, \Lambda^\nu{}_\kappa = \begin{pmatrix} \gamma & 0 & 0 & \beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \beta\gamma & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix} \begin{pmatrix} \gamma & 0 & 0 & \beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \beta\gamma & 0 & 0 & \gamma \end{pmatrix} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$$

using $\gamma^2(1-\beta^2) = 1$.

So $p\cdot q \to p'\cdot q' = p\cdot q$ is invariant under boosts.

Lorentz transformations: group of matrices that preserves the scalar product between 4-vectors

Metric tensor $g_{\mu\nu}$ is invariant

$$g_{\mu\nu} \to \Lambda^{\lambda}{}_{\mu} \Lambda^{\kappa}{}_{\nu} \, g_{\lambda\kappa} = g_{\mu\nu}$$

$$g^{\mu\nu} \to \Lambda^{\mu}{}_{\lambda} \Lambda^{\nu}{}_{\kappa} \, g^{\lambda\kappa} = g^{\mu\nu}$$

Note: alternate sign convention $g_{\mu\nu} = \mathrm{diag}(-1, +1, +1, +1)$

The mass (rest mass) of a particle is $p^2 = m^2 = E^2 - |\underline{p}|^2$ is invariant under Lorentz transformations.

For a photon, $m = 0$ so $p^2 = E^2 - |\underline{p}|^2 = 0$.
In any frame the speed of light is $|\underline{p}|/E = 1$.