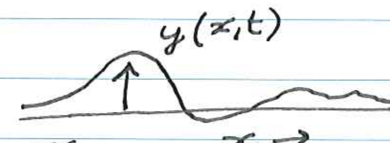
III. Klein-Gordon fields

What is field theory? Field is a function $\phi(x)$ that is a function of spacetime x^μ.

Familiar example: electric field $\vec{E}(t, \underline{x})$. $\vec{E}$ is a (three-) vector field; it has magnitude & direction as function of x^μ. The equation of motion for $\vec{E}$ is Maxwell's equations.

Classical field theory



Simple example: waves on an infinite string with string tension T and density $\rho = \frac{\text{mass}}{\text{length}}$.

$y(x,t)$ is vertical displacement of string.

Satisfies wave equation: $\dfrac{\partial^2 y}{\partial t^2} = v^2 \dfrac{\partial^2 y}{\partial x^2}$ eqn of motion.

where $v^2 = T/\rho$ is wave velocity.

Equation of motion obtained by "principle of least action"
$y(x,t)$ evolves from $t = t_1$ to $t = t_2$ in such a way to minimize the action S.

Action $S = \int dt\, L$ where L = Lagrangian.

For infinite string $L = \displaystyle\int_{-\infty}^{\infty} dx\ \frac{1}{2}\left[\left(\frac{\partial y}{\partial t}\right)^2 - v^2\left(\frac{\partial y}{\partial x}\right)^2\right]$

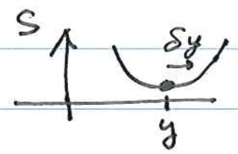
Thus we can also write $S = \int dt\, dx\, \mathcal{L}(y, \dot{y}, y')$

where $\mathcal{L}$ is the Lagrangian density (or just Lagrangian) and it depends on y, $\dot{y} = \frac{\partial y}{\partial t}$, and $y' = \frac{\partial y}{\partial x}$.

The equation of motion minimizes the action:

Consider perturbing the solution
$$y \to y + \delta y$$
$$\dot{y} \to \dot{y} + \delta \dot{y}$$
$$y' \to y' + \delta y'$$
with boundary conditions fixed.

S should be invariant at 1st order in $\delta y, \delta \dot{y}, \delta y'$.



so $S \to S + \delta S = S$ at 1st order

$$\mathcal{L}(y, \dot{y}, y') \to \mathcal{L}(y + \delta y, \dot{y} + \delta \dot{y}, y' + \delta y')$$
$$= \mathcal{L} + \frac{\partial \mathcal{L}}{\partial y}\delta y + \frac{\partial \mathcal{L}}{\partial \dot{y}}\delta \dot{y} + \frac{\partial \mathcal{L}}{\partial y'}\delta y'$$

$$S \to \int dt\, dx \left(\mathcal{L} + \frac{\partial \mathcal{L}}{\partial y}\delta y + \frac{\partial \mathcal{L}}{\partial \dot{y}}\delta \dot{y} + \frac{\partial \mathcal{L}}{\partial y'}\delta y' \right)$$

$$= S + \delta S$$

$$\delta S = \int dt\, dx \left(\frac{\partial \mathcal{L}}{\partial y}\delta y + \frac{\partial \mathcal{L}}{\partial \dot{y}}\delta\left(\frac{\partial y}{\partial t}\right) + \frac{\partial \mathcal{L}}{\partial y'}\delta\left(\frac{\partial y}{\partial x}\right) \right)$$

$$= \int dt\, dx \left(\frac{\partial \mathcal{L}}{\partial y}\delta y - \frac{\partial}{\partial t}\frac{\partial \mathcal{L}}{\partial \dot{y}}\delta y - \frac{\partial}{\partial x}\frac{\partial \mathcal{L}}{\partial y'}\delta y \right)$$

+ boundary terms

$$\delta S = \int dt\, dx \left(\frac{\partial \mathcal{L}}{\partial y} - \frac{\partial}{\partial t} \frac{\partial \mathcal{L}}{\partial \dot{y}} - \frac{\partial}{\partial x} \frac{\partial \mathcal{L}}{\partial y'} \right) \delta y = 0$$

for any $\delta y(x,t)$ provided the Euler-Lagrange equation is satisfied:

$$\frac{\partial}{\partial t} \frac{\partial \mathcal{L}}{\partial \dot{y}} + \frac{\partial}{\partial x} \frac{\partial \mathcal{L}}{\partial y'} - \frac{\partial \mathcal{L}}{\partial y} = 0$$

$$\frac{\partial}{\partial t} \dot{y} - v^2 \frac{\partial}{\partial x} y' = \frac{\partial^2 y}{\partial t^2} - v^2 \frac{\partial^2 y}{\partial x^2} = 0$$

Wave equation.

Lagrangian $\mathcal{L}$ defines the theory.

<u>Klein-Gordon field theory</u>

The simplest kind of field is a real, scalar field $\phi(x)$.
i.e. ϕ is a real number (no direction, no complex phase)

The Lagrangian is $\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2} m^2 \phi^2$

Note: $(\partial_\mu \phi)^2$ is sloppy notation for $(\partial_\mu \phi)(\partial^\mu \phi) = g_{\mu\nu}(\partial^\mu \phi)(\partial^\nu \phi)$

So $\mathcal{L} = \frac{1}{2}\left(\frac{\partial \phi}{\partial t}\right)^2 - \frac{1}{2}\left(\frac{\partial \phi}{\partial x}\right)^2 - \frac{1}{2}\left(\frac{\partial \phi}{\partial y}\right)^2 - \frac{1}{2}\left(\frac{\partial \phi}{\partial z}\right)^2 - \frac{1}{2} m^2 \phi^2$

$= \frac{1}{2}\dot{\phi}^2 - \frac{1}{2}|\nabla \phi|^2 - \frac{1}{2} m^2 \phi^2$

And the action is $S = \int dt \int d^3x\, \mathcal{L} = \int d^4x\, \mathcal{L}$

Equation of motion: minimize the action under variation

$$\phi(x) \rightarrow \phi(x) + \delta\phi(x)$$

$$\partial_\mu \phi \rightarrow \partial_\mu \phi + \delta(\partial_\mu \phi)$$

$$\mathcal{L}(\phi, \partial_\mu\phi) \rightarrow \mathcal{L}(\phi + \delta\phi, \partial_\mu\phi + \delta(\partial_\mu\phi))$$

$$= \mathcal{L}(\phi, \partial_\mu\phi) + \frac{\partial \mathcal{L}}{\partial \phi}\delta\phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\delta(\partial_\mu\phi)$$

So the action is:

$$S \rightarrow S + \delta S = \int d^4x \left(\mathcal{L}(\phi, \partial_\mu\phi) + \frac{\partial \mathcal{L}}{\partial \phi}\delta\phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\delta(\partial_\mu\phi) \right)$$

$$\delta S = \int d^4x \left(\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu\phi)} \right) \right) \delta\phi$$

So the Euler-Lagrange equation is

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu\phi)} \right) - \frac{\partial \mathcal{L}}{\partial \phi} = 0$$

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu\phi)} = \frac{\partial}{\partial(\partial_\mu\phi)} \left(\frac{1}{2}(\partial_\nu\phi)(\partial^\nu\phi) + \frac{1}{2}m^2\phi^2 \right)$$

$$= \frac{1}{2}\partial^\mu\phi \times 2 = \partial^\mu\phi$$

$$\Rightarrow \quad \partial_\mu\partial^\mu\phi + m^2\phi = (\partial^2 + m^2)\phi = 0$$

Klein-Gordon equation.

$$\left(\frac{\partial^2}{\partial t^2} - |\nabla|^2 + m^2 \right) \phi(x) = 0$$

Consider Fourier transforming $\phi(x)$:

$$\phi(x) = \int \frac{d\omega}{2\pi} e^{-i\omega t} \int \frac{d^3k}{(2\pi)^3} e^{i\underline{k}\cdot\underline{x}} \tilde{\phi}(\omega, \underline{k})$$

Then acting on $\phi(x)$ with $(\partial^2 + m^2)$ KG operator:

$$\int \frac{d\omega}{2\pi} \int \frac{d^3k}{(2\pi)^3} (-\omega^2 + |\underline{k}|^2 + m^2)\, e^{-i\omega t + i\underline{k}\cdot\underline{x}} \tilde{\phi}(\omega, \underline{k}) = 0$$

So $\omega^2 - |\underline{k}|^2 - m^2 = 0$. Quantum mechanics tells us that $E = \hbar\omega = \omega$ and $p = \hbar k = k$. So KG eqn. is equivalent to $E^2 - |\underline{p}|^2 = m^2 = p^2$.

Hamiltonian

Recall from classical mechanics: for a dynamical variable q, define a conjugate momentum $p = \frac{\partial L}{\partial \dot{q}}$. The Hamiltonian is $H = p\dot{q} - L$.

Or if there are many dynamical variables q_i (labeled by i), then $p_i = \frac{\partial L}{\partial \dot{q}_i}$ and $H = \sum_i p_i \dot{q}_i - L$.

example: 3-D SHO $\quad L = \frac{1}{2} m |\dot{\underline{x}}|^2 - \frac{1}{2} k |\underline{x}|^2$

$$(q_1, q_2, q_3) = (x_1, x_2, x_3) = \underline{x} \quad \text{dynamical variables}$$

$$\underline{p} = (p_1, p_2, p_3) = m(\dot{x}_1, \dot{x}_2, \dot{x}_3) = m\dot{\underline{x}} \quad \text{canonical momentum (same as usual mom.)}$$

$$H = \underline{p}\cdot\dot{\underline{x}} - L = \underbrace{\tfrac{1}{2} m |\dot{\underline{x}}|^2}_{KE} + \underbrace{\tfrac{1}{2} k |\underline{x}|^2}_{PE} \quad \text{(Energy)}$$

example: infinite string $L = \int dx \, \frac{1}{2}\left(\left(\frac{\partial y}{\partial t}\right)^2 - v^2 \left(\frac{\partial y}{\partial x}\right)^2 \right)$

$\rho(x) =$

Here dynamical variable is $y(x)$, ~~a continuous~~ labeled by a continuous variable x (insted of discrete index i).

Treat x as discrete variable:

$$L = \sum_x dx \, \frac{1}{2}\left(\left(\frac{\partial y}{\partial t}\right)^2 - v^2 \left(\frac{\partial y}{\partial x}\right)^2 \right)$$

Canonical momentum

$$p(x) = \frac{\partial L}{\partial \dot{y}(x)} = \frac{\partial}{\partial \dot{y}(x)} \sum_{x'} dx' \, \frac{1}{2}\left(\dot{y}(x')^2 - v^2 y'(x')^2 \right)$$

$$= dx \, \dot{y}(x)$$

Useful to define $p(x) = \pi(x)\, dx$, where $\pi(x)$ is

$$\pi(x) = \frac{\partial \mathcal{L}}{\partial \dot{y}(x)} = \text{canonical momentum density}$$

Hamiltonian is:

$$H = \sum_x \dot{y}(x)\, p(x) - L$$

$$= \int dx \left(\dot{y}(x)\, \pi(x) - \mathcal{L} \right) = \int dx \, \mathcal{H}$$

Define Hamiltonian density $\mathcal{H} = \dot{y}(x)\, \pi(x) - \mathcal{L}$

$$\mathcal{H} = \underbrace{\frac{1}{2}\left(\frac{\partial y}{\partial t}\right)^2}_{\text{Kinetic energy density}} + \underbrace{\frac{1}{2} v^2 \left(\frac{\partial y}{\partial x}\right)^2}_{\text{tension term}} = \text{energy density (per unit length)}$$

Klein-Gordon ~~equation~~ field:

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2} m^2 \phi^2$$

dynamical variable $q(x) = y(x)$

canonical momentum $\pi(x) = \dfrac{\partial \mathcal{L}}{\partial \dot{\phi}(x)} = \dot{\phi}(x)$

Hamiltonian (density)

$$\mathcal{H} = \dot{\phi}(x)\pi(x) - \mathcal{L}(x)$$
$$= \underbrace{\frac{1}{2}\dot{\phi}(x)^2}_{\text{KE of field}} + \underbrace{\frac{1}{2}|\nabla\phi|^2}_{\substack{\text{gradient (tension)}\\ \text{Energy}}} + \underbrace{\frac{1}{2} m^2 \phi^2}_{\substack{\text{rest energy}\\ \text{of field.}}}$$

energy density

$$H = \int d^3x\, \mathcal{H}(x) = \text{total energy.}$$

~~Quantization of Klein Gordon~~

Noether's theorem & symmetry

Important connection between (continuous) transformations, symmetry, and conservation laws.

Transformations: two classes

(1) continuous transformations: parametrized by continuous variable

e.g. rotations in 2D: $\underset{(x,y)}{\underline{x}} \to \underset{(x',y')}{\underline{x}'} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \underline{x}$

Infinitesimal transformation: take parameter $\ll 1$

e.g. $\underline{x} \to \underline{x}' = \begin{pmatrix} 1 & -\theta \\ \theta & 1 \end{pmatrix} \underline{x} = \underline{x} + \theta \underline{\Delta x}$

where $\underline{\Delta x} = (-y, x)$

(2) Discrete transformation: no continuous parameter

e.g. parity $\underline{x} \to -\underline{x}$

Consider infinitesimal transformation acting on field $\phi(x)$:

$$\phi(x) \to \phi'(x) = \phi(x) + \alpha \Delta\phi(x)$$

where $\alpha \ll 1$ is infinitesimal parameter.

Transformation is a symmetry if the equations of motion are unchanged. implies $\mathcal{L} \to \mathcal{L}' = \mathcal{L}$ upto a total derivative.

So want $\mathcal{L} \to \mathcal{L}' = \mathcal{L} + \alpha \underbrace{\partial_\mu \mathcal{J}^\mu}_{\text{total derivative}}$

The action is $S \to S' = S + \underbrace{\alpha \int d^4x \; \partial_\mu \mathcal{J}^\mu}$

surfaceterm doesn't impact eqn of motion since assumed variation $\delta\phi$ vanished on boundary of integration.

Consider evaluating $\mathcal{L}'$ explicitly by $\phi \to \phi'$.

$$\mathcal{L} \to \mathcal{L}' = \mathcal{L} + \frac{\partial \mathcal{L}}{\partial \phi} \alpha \Delta\phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_\mu (\alpha \Delta \phi)$$

$$= \mathcal{L} + \underbrace{\left(\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right)}_{=0 \text{ by eqn of motion}} \alpha \Delta\phi + \alpha \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \Delta\phi \right)$$

So we can set $\partial_\mu \mathcal{J}^\mu = \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \Delta\phi \right)$

or $\boxed{\partial_\mu j^\mu = 0 \quad \text{where} \quad j^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \Delta\phi - \mathcal{J}^\mu}$

The quanity j^μ is a current 4-vector.

j^0 is the number density of a conserved charge

$\vec{j}$ is the 3-current for that charge

Total charge: $Q = \int d^3x \, j^0$

$$\frac{dQ}{dt} = \int_{\substack{\text{all}\\\text{space}}} d^3x \, \frac{dj^0}{dt} = -\int d^3x \, (\nabla \cdot \underline{j}) = -\int_{\infty} \underline{j} \cdot d\underline{S} = 0$$

using $\partial_\mu j^\mu = \frac{dj^0}{dt} + \nabla \cdot \underline{j} = 0$

assuming $\underline{j}$ vanishes at ∞.

charge conservation.

$\partial_\mu j^\mu = \frac{\partial j^0}{\partial t} + \underline{\nabla} \cdot \underline{j} = 0$ is the continuity equation.

local change in number density j^0 caused by a current of charge flowing from that position.

example: shift symmetry $\phi \to \phi + \alpha$

massless scalar field: $\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2$

symmetry since $\mathcal{L} \to \mathcal{L}$.

Then $j^\mu = \partial^\mu \phi$ is conserved current.

note $\partial_\mu j^\mu = \partial^2 \phi = 0$ by EoM,

example: two degenerate ~~complex~~ scalar field ϕ_1, ϕ_2

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi_1)^2 - \frac{1}{2}m^2\phi_1^2 + \frac{1}{2}(\partial_\mu \phi_2)^2 - \frac{1}{2}m^2\phi_2^2$$

symmetry under field redefinition:

$$\phi_1 \to \phi_1' = \phi_1 - \alpha\phi_2$$

$$\phi_2 \to \phi_2' = \phi_2 + \alpha\phi_1$$

$$\begin{pmatrix}\phi_1\\\phi_2\end{pmatrix} \to \begin{pmatrix}\phi_1'\\\phi_2'\end{pmatrix} = \begin{pmatrix}\cos\alpha & -\sin\alpha\\\sin\alpha & \cos\alpha\end{pmatrix}\begin{pmatrix}\phi_1\\\phi_2\end{pmatrix}$$

$$\mathcal{L} \to \mathcal{L}' = \frac{1}{2}(\partial_\mu \phi_1)^2 - \alpha \cancel{(\partial_\mu \phi_1)(\partial^\mu \phi_2)} - \frac{1}{2} m^2 \phi_1^2 + \cancel{m^2 \phi_1 \phi_2 \alpha}$$
$$+ \frac{1}{2}(\partial_\mu \phi_2)^2 + \alpha \cancel{(\partial_\mu \phi_1)(\partial^\mu \phi_2)} - \frac{1}{2} m^2 \phi_2^2 - \cancel{m^2 \phi_1 \phi_2 \alpha}$$
$$= \mathcal{L}$$

Then $$j^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_1)} \Delta\phi_1 + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_2)} \Delta\phi_2$$

$$= -(\partial^\mu \phi_1)\phi_2 + (\partial^\mu \phi_2)\phi_1$$

$$= \phi_1 \overleftrightarrow{\partial^\mu} \phi_2 = \phi_1(\partial^\mu \phi_2) - (\partial^\mu \phi_1)\phi_2$$

Note: $\partial_\mu j^\mu = \phi_1 \partial^2 \phi_2 - (\partial^2 \phi_1)\phi_2 = 0$ by E.o.M.

Can also express this theory as a single complex scalar field

Define $\phi(x) = \dfrac{\phi_1 + i\phi_2}{\sqrt{2}}$

$$\mathcal{L} = |\partial_\mu \phi|^2 - m^2 |\phi|^2 = \partial_\mu \phi \, \partial^\mu \phi^* - m^2 \phi^* \phi$$

Symmetry: $\phi \to e^{i\alpha}\phi = \phi + i\alpha\phi$ (for $\alpha \ll 1$)
$\phi^* \to e^{-i\alpha}\phi^* = \phi^* - i\alpha\phi^*$

So $\Delta\phi = i\phi$, $\Delta\phi^* = -i\phi^*$

$$j^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \Delta\phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^*)} \Delta\phi^*$$

$$= i(\partial^\mu \phi^*)\phi - i(\partial^\mu \phi)\phi = -i\phi^* \overleftrightarrow{\partial^\mu} \phi$$

Theory of two real scalars with rotational field redef. symmetry equivalent to theory of one complex scalar field with a rephasing symmetry.

Preceding examples were <u>field transformations</u>.

Also can have <u>space time transformations</u>:

e.g. Lorentz transform $x^\mu \to x'^\mu = \Lambda^\mu{}_\nu x^\nu$

Translations $x^\mu \to x'^\mu = x^\mu + a^\mu$

$\Big\}$ Poincare transform

Infinitesimal form: $x^\mu \to x'^\mu = x^\mu + \alpha \Delta x^\mu$

Spacetime transformations can be expressed as field transformation: $\phi(x) \to \phi'(x)$

note: $\phi(x) = \phi'(x')$

new field at new coord = old field at old coord

Then $\phi(x - \alpha\Delta x) = \phi'(x)$

$$\phi(x) \to \phi'(x) = \phi(x - \alpha\Delta x) = \phi(x) - \alpha\, \partial_\mu \phi\, \Delta x^\mu$$

Then $$\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \Delta\phi = (\partial^\mu \phi)(-\partial_\nu \phi \Delta x^\nu)$$

Can also apply transformation to $\mathcal{L}$ directly (also a scalar)

$$\mathcal{L}(x) \to \mathcal{L}'(x) = \mathcal{L}(x) - \alpha\, \partial_\mu \underbrace{\mathcal{L}\, \Delta x^\mu}_{-\mathcal{J}^\mu}$$

So $$j^\mu = -(\partial^\mu\phi)(\partial_\nu \phi \Delta x^\nu) + \mathcal{L}\Delta x^\mu$$

e.g. translations: set $\Delta x^\mu = -a^\mu$

$$j^\mu = (\partial^\mu \phi\, \partial_\nu \phi)\, a^\nu - \mathcal{L} g^\mu{}_\nu\, a^\nu$$

$$= (\partial^\mu \phi\, \partial_\nu \phi - \mathcal{L} g^\mu{}_\nu)\, a^\nu$$

j^μ is conserved for any a^ν, so there are actually four separate conserved currents ($\nu = 0, 1, 2, 3$).

Can write $j^\mu = T^\mu{}_\nu\, a^\nu$

where $$T^\mu{}_\nu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_\nu \phi - \mathcal{L} g^\mu{}_\nu$$

$$= \partial^\mu \phi\, \partial_\nu \phi - \mathcal{L} g^\mu{}_\nu$$

$T^{\mu\nu}$ is the energy-momentum tensor:

$$T^{00} = \dot{\phi}^2 - \mathcal{L} = \mathcal{H} = \text{energy density}$$

conserved charge is $H = \int d^3x\, \mathcal{H}$ = energy associated with time translations $x \to (t+a, \underline{x})$

$$T^{0i} = \dot{\phi}\, \partial^i \phi - 0 = -\dot{\phi}\, \partial_i \phi$$

conserved charge is $\underline{P} = -\int d^3x\, \dot{\phi}\, \underline{\nabla} \phi$ is physical momentum ~~associated~~ carried by the field.

remaining T^{ij} components describe stress (momentum flux)

Also called stress-energy tensor.

Quantization of Klein-Gordon fields

Classical theory: coordinate q_i
canonical momentum p_i

Quantum theory: promote q_i, p_i to QM operators satisfying commutation relations.

$$[q_i, p_j] = i\delta_{ij} \quad , \quad [q_i, q_j] = [p_i, p_j] = 0$$

KG fields in 0 space-dim. + 1 time dim.:

$$\mathcal{L} = \frac{1}{2}\dot{\phi}^2 - \frac{1}{2}m^2\phi^2 = L$$
$$\mathcal{H} = \frac{1}{2}\dot{\phi}^2 + \frac{1}{2}m^2\phi^2 = H$$

Simple harmonic oscillator with frequency m.

Commutation relations: $q = \phi$, $p = \dot{\phi}$

$$[\phi, \dot{\phi}] = i \quad , \quad [\phi, \phi] = [\dot{\phi}, \dot{\phi}] = 0$$

Diagonalize Hamiltonian using ladder operators

$$\phi = \frac{1}{\sqrt{2m}}(a + a^\dagger) \quad , \quad \dot{\phi} = -i\sqrt{\frac{m}{2}}(a - a^\dagger)$$

$$\Rightarrow H = m\left(a^\dagger a + \frac{1}{2}\right)$$

Eigenstates of Hamiltonian are $|n\rangle = (a^\dagger)^n |0\rangle$

with energy $E_n = m\left(n + \frac{1}{2}\right)$

n = # of particles with mass m.

vacuum energy $= E_0 = \frac{m}{2}$

field operator creates a particle from vacuum

$$\phi|0\rangle = \frac{1}{\sqrt{2m}}|1\rangle$$

KG fields in 3 (space) + 1 (time) dimension:

$$q_i \to \phi(\underline{x}) \qquad p_i \to \pi(\underline{x})\, d^3x \qquad (\pi(\underline{x}) = \dot{\phi}(\underline{x}))$$

We are working in Schrödinger picture where operators are time-independent $(\phi(\underline{x}) = \phi(t=0, \underline{x}))$

$$[\phi(\underline{x}), \pi(\underline{x}')\, d^3x'] = i\delta_{\underline{x},\underline{x}'} = i\delta^3(\underline{x}-\underline{x}')\, d^3x'$$

So canonical commutation relations are:

$$\boxed{\begin{aligned} &[\phi(\underline{x}), \pi(\underline{x}')] = i\delta^3(\underline{x}-\underline{x}') \\ &[\phi(\underline{x}), \phi(\underline{x}')] = [\pi(\underline{x}), \pi(\underline{x}')] = 0 \end{aligned}}$$

Generalization to multiple scalar fields:

$$\boxed{\begin{aligned} &[\phi_i(\underline{x}), \pi_j(\underline{x}')] = i\,\delta^3(\underline{x}-\underline{x}')\,\delta_{ij} \\ &[\phi_i(\underline{x}), \phi_j(\underline{x}')] = [\pi_i(\underline{x}), \pi_j(\underline{x}')] = 0 \end{aligned}}$$

Just like SHO, we can diagonalize the Hamiltonian using ladder operators. First, Fourier transform:

$$\phi(\underline{x}) = \int \frac{d^3p}{(2\pi)^3} e^{i\underline{p}\cdot\underline{x}} \tilde{\phi}(\underline{p})$$

Then expand $\tilde{\phi}(\underline{p}) = \frac{1}{\sqrt{2\omega_p}} (a_{\underline{p}} + a^\dagger_{-\underline{p}})$ using ladder ops.

Similarly: $\pi(\underline{x}) = \int \frac{d^3p}{(2\pi)^3} e^{i\underline{p}\cdot\underline{x}} \tilde{\pi}(\underline{p})$

$$= \int \frac{d^3p}{(2\pi)^3} e^{i\underline{p}\cdot\underline{x}} (-i) \sqrt{\frac{\omega_p}{2}} (a_{\underline{p}} - a^\dagger_{-\underline{p}})$$

Each Fourier mode (with momentum $\underline{p}$) is treated as independent SHO with frequency $\omega_{\underline{p}} = \sqrt{|\underline{p}|^2 + m^2}$.

$\phi(\underline{x})$ is a real field: $\phi(\underline{x}) = \phi^*(\underline{x})$ for a classical field

$\Rightarrow$ QM field satisfies $\phi(\underline{x}) = \phi^\dagger(\underline{x})$ (Hermitian conjugate)

$$\boxed{\begin{aligned} \phi(\underline{x}) &= \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_p}} (a_{\underline{p}} e^{i\underline{p}\cdot\underline{x}} + a^\dagger_{\underline{p}} e^{-i\underline{p}\cdot\underline{x}}) = \phi^\dagger(\underline{x}) \\ \pi(\underline{x}) = \dot{\phi}(\underline{x}) &= \int \frac{d^3p}{(2\pi)^3} \sqrt{\frac{\omega_p}{2}} (-i) (a_{\underline{p}} e^{i\underline{p}\cdot\underline{x}} - a^\dagger_{\underline{p}} e^{-i\underline{p}\cdot\underline{x}}) = \pi^\dagger(\underline{x}) \end{aligned}}$$

Can invert and solve for $a_{\underline{p}}, a^\dagger_{\underline{p}}$ in terms of ϕ, π:

$$a_{\underline{p}} = \int d^3x \, e^{-i\underline{p}\cdot\underline{x}} \left(\sqrt{\frac{\omega_p}{2}} \phi(\underline{x}) + i \sqrt{\frac{1}{2\omega_p}} \pi(\underline{x}) \right)$$

$$a^\dagger_{\underline{p}} = \int d^3x \, e^{-i\underline{p}\cdot\underline{x}} \left(\sqrt{\frac{\omega_p}{2}} \phi(\underline{x}) - i \sqrt{\frac{1}{2\omega_p}} \pi(\underline{x}) \right)$$

$$\Rightarrow \boxed{\begin{aligned} [a_{\underline{p}}, a^\dagger_{\underline{p}'}] &= (2\pi)^3 \delta^3(\underline{p} - \underline{p}') \\ [a_{\underline{p}}, a_{\underline{p}'}] &= [a^\dagger_{\underline{p}}, a^\dagger_{\underline{p}'}] = 0 \end{aligned}}$$

Can check:

$$[\phi(\underline{x}), \pi(\underline{x}')] = \int \frac{d^3p}{(2\pi)^3} \int \frac{d^3p'}{(2\pi)^3} \left(\frac{-i}{2}\right) [a_{\underline{p}} + a^{\dagger}_{-\underline{p}}, a_{\underline{p}'} - a^{\dagger}_{-\underline{p}'}] \times e^{i(\underline{p}\cdot\underline{x} + \underline{p}'\cdot\underline{x}')}$$

$$= \frac{-i}{2} \int \frac{d^3p}{(2\pi)^3} \times (-2) \times e^{i\underline{p}\cdot(\underline{x} - \underline{x}')}$$

$$= i\,\delta^3(\underline{x} - \underline{x}')$$

using $\quad \delta^3(\underline{x} - \underline{x}') = \int \frac{d^3p}{(2\pi)^3} e^{i\underline{p}\cdot(\underline{x} - \underline{x}')}$

Evaluate Hamiltonian:

$$H = \int d^3x\, \mathcal{H} = \int d^3x \left(\frac{1}{2}\pi^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi^2 \right)$$

$$\neq \cancel{\int d^3x \int \frac{d^3p}{(2\pi)^3} \int \frac{d^3p'}{(2\pi)^3} \frac{1}{2} \left\{ -\frac{\omega_p^2}{2} (a_{\underline{p}} - a^{\dagger}_{\underline{p}})(a_{\underline{p}} - a^{\dagger}_{\underline{p}}) \right.}$$

$$\cancel{+ \frac{1}{2\omega_p}(i\underline{p})^2 (a_{\underline{p}} + a^{\dagger}_{\underline{p}})(a_{\underline{p}} + a^{\dagger}_{\underline{p}})}$$

$$\cancel{+ \frac{m^2}{2\omega_p} (a_{\underline{p}} +}$$

$$= \int d^3x \int \frac{d^3p}{(2\pi)^3} \int \frac{d^3p'}{(2\pi)^3} e^{i(\underline{p}\cdot\underline{x} + \underline{p}'\cdot\underline{x})} \frac{1}{2}$$

$$\times \left(-\frac{\sqrt{\omega_p \omega_{p'}}}{2} (a_{\underline{p}} - a^{\dagger}_{\underline{p}})(a_{\underline{p}'} - a^{\dagger}_{-\underline{p}'}) \right.$$

$$- \frac{\underline{p}\cdot\underline{p}'}{2\sqrt{\omega_p \omega_{p'}}} (a_{\underline{p}} + a^{\dagger}_{-\underline{p}})(a_{\underline{p}'} + a^{\dagger}_{-\underline{p}'})$$

$$\left. + \frac{m^2}{2\sqrt{\omega_p \omega_{p'}}} (a_{\underline{p}} + a^{\dagger}_{-\underline{p}})(a_{\underline{p}'} + a^{\dagger}_{-\underline{p}}) \right)$$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2} \Big(-\frac{\omega_p}{2} (a_p - a^\dagger_{-p})(a_{-p} - a^\dagger_p)$$

$$+ \frac{\omega_p}{2} (a_p + a^\dagger_{-p})(a_{-p} + a^\dagger_p) \Big)$$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{\omega_p}{4} \Big(-\cancel{a_p a_{-p}} + a_p a_p^\dagger + a^\dagger_{-p} a_{-p} - \cancel{a^\dagger_{-p} a^\dagger_p}$$

$$+ \cancel{a_p a_{-p}} + a_p a_p^\dagger + a^\dagger_{-p} a_{-p} + \cancel{a^\dagger_{-p} a^\dagger_p} \Big)$$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{\omega_p}{2} \left(a_p a_p^\dagger + a_p^\dagger a_p \right)$$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{\omega_p}{2} \left(2 a_p^\dagger a_p + [a_p, a_p^\dagger] \right)$$

$$= \int \frac{d^3p}{(2\pi)^3} \omega_p \Big(a_p^\dagger a_p + \frac{1}{2} \underbrace{[a_p, a_p^\dagger]}_{= (2\pi)^3 \delta^3(0) = \infty} \Big)$$

usual number operator counting particles with momentum mode p.

Infinite constant.
Zero point vacuum energy of infinite number of SHO's.
Discard infinite constant since physics depends on energy differences.

~~a_p creates a particle with momentum p~~

Quantum mechanical states:

Interpret $a_p^\dagger |0\rangle$ as a one particle state with momentum p and energy $\omega_p = \sqrt{|p|^2 + m^2} = E_p$

Vacuum: $|0\rangle$ such that $a_p |0\rangle = 0$.

$$H \left(a_p^\dagger |0\rangle \right) = \int \frac{d^3p'}{(2\pi)^3} \omega_{p'} a^\dagger_{p'} a_{p'} a_p^\dagger |0\rangle = \int \frac{d^3p'}{(2\pi)^3} \omega_{p'} a^\dagger_{p'} \left(a_p^\dagger a_{p'} + (2\pi)^3 \delta^3(p - p') \right) |0\rangle$$

$$= \omega_p \left(a_p^\dagger |0\rangle \right)$$

Think of $(2\pi)^3 \delta^3(0)$ = volume of space (infinite)

$$\text{Volume} = \int d^3x = \lim_{p\to 0} \int d^3x \, e^{ip\cdot x} = \lim_{p\to 0} \delta^{(3)}(p) = \delta^3(0)$$

So infinite constant is $\left(\int \frac{d^3p}{(2\pi)^3} \frac{\omega_p}{2}\right) \times$ volume

Energy density of vacuum:

$$\rho_0 = \frac{E_0}{\text{Volume}} = \int \frac{d^3p}{(2\pi)^3} \frac{\omega_p}{2} = \infty$$

Zero-point energy density → still diverges.

Introduce a cut-off: new physics at scale $\omega_p = \Lambda = 1$ TeV to cancel out divergence

$$\Rightarrow \rho_0 \sim \Lambda^4 = 1 \text{ TeV}^4 = 10^{48} \text{ eV}^4$$

observed vacuum energy density (dark energy (cosmological const)

$$\rho_0^{obs} \approx 10^{-11} \text{ eV}^4$$

cosmological constant problem $\rho_0^{obs} \ll \rho_0(\Lambda)$
and if $\rho_0 \gtrsim \rho_0^{obs}$ then universe becomes dark energy dominated before galaxies form.
→ no galaxies formed.

momentum operator: $\underline{P} = -\int d^3x\, \pi(\underline{x})\, \nabla\phi(\underline{x}) = \int \frac{d^3p}{(2\pi)^3}\, \underline{p}\, a_{\underline{p}}^\dagger a_{\underline{p}}$

$$\underline{P}\left(a_{\underline{p}}^\dagger |0\rangle\right) = \underline{p}\left(a_{\underline{p}}^\dagger |0\rangle\right)$$

Multiparticle state:

$a_{\underline{p}}^\dagger a_{\underline{q}}^\dagger |0\rangle$ = particle 1 with momentum $\underline{p}$
particle 2 with momtum $\underline{q}$

$= a_{\underline{q}}^\dagger a_{\underline{p}}^\dagger |0\rangle$ since $a_{\underline{p}}^\dagger, a_{\underline{q}}^\dagger$ commute

Particles described by Klein-Gordon fields are bosons that obey Bose-Einstein statistics (state symmetric under exchange)

Note: $a_{\underline{p}}^\dagger a_{\underline{p}}^\dagger |0\rangle$ is a valid state with two particles in the same state (momentum $\underline{p}$)
No exclusion principle for bosons.

Normalization of states: vacuum $|0\rangle$: normalized $\langle 0|0\rangle = 1$.

$|\underline{p}\rangle = \sqrt{2E_p}\, a_{\underline{p}}^\dagger |0\rangle$ one-particle state with mom. $\underline{p}$.

$$\langle \underline{p} | \underline{q} \rangle = \sqrt{4E_pE_q}\, \langle 0| a_{\underline{p}} a_{\underline{q}}^\dagger |0\rangle$$

$$= 2\sqrt{E_pE_q}\, \langle 0 | a_{\underline{q}}^\dagger a_{\underline{p}} + (2\pi)^3 \delta^3(\underline{p}-\underline{q}) |0\rangle$$

$$= (2E_p)(2\pi)^3 \delta^3(\underline{p}-\underline{q})$$

Normalization factor $\sqrt{2E_p}$ useful so that $\langle p|q\rangle$ is Lorentz invariant.

Boost along $\hat{z}$ direction:

$$\langle p|q\rangle \rightarrow \langle p'|q'\rangle = (2E_{p'})(2\pi)^3\,\delta^3(p'-q')$$

where
$$p \rightarrow p' = (p_x, p_y, \gamma p_z + \beta\gamma E_p)$$
$$q \rightarrow q' = (q_x, q_y, \gamma q_z + \gamma\beta E_q)$$

$$E_p \rightarrow E_p' = \gamma E_p + \beta\gamma p_z \qquad \text{(same for } E_q\text{)}$$

$$\delta^3(p'-q') = \delta(p_x-q_x)\,\delta(p_y-q_y)\,\delta(p_z'-q_z')$$

note:
$$p_z'-q_z' = \gamma(p_z-q_z) + \beta\gamma(E_p-E_q)$$
$$= \gamma(p_z-q_z) + \beta\gamma\frac{\partial E_p}{\partial p_z}(p_z-q_z)$$
$$= \left(\gamma + \gamma\beta\frac{\partial E_p}{\partial p_z}\right)(p_z-q_z) \qquad \frac{\partial E_p}{\partial p_z} = \frac{p_z}{E_p}$$

use $\delta(ax) = \frac{1}{|a|}\delta(x)$

$$\delta(p_z'-q_z') = \frac{1}{\gamma + \gamma\beta\left(\frac{\partial E_p}{\partial p_z}\right)}\,\delta(p_z-q_z)$$
$$= \frac{E_p}{\gamma E_p + \gamma\beta p_z}\,\delta(p_z-q_z) = \frac{E_p}{E_p'}\,\delta(p_z-q_z)$$

So
$$\langle p|q\rangle \rightarrow \langle p'|q'\rangle = (2E_{p'})(2\pi)^3\cdot\frac{E_p}{E_{p'}}\,\delta^3(p-q) = \langle p|q\rangle$$

(Normalization without $\sqrt{2E_p}$ also used esp. in nonrelativistic context since $\sqrt{2E_p} \simeq \sqrt{2m} = \text{const}$).

Completeness relation: $\mathbb{1} = \sum_{\text{states } n} |n\rangle\langle n|$ usual relation

Here need to divide by $\frac{1}{2E_p}$:

$$(\mathbb{1})_{\substack{\text{one-particle}\\ \text{states only}}} = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} |p\rangle\langle p|$$

The integral $\int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p}$ comes up often when summing over states. ~~called Lorentz invariant~~
It is Lorentz invariant; called Lorentz-invariant phase space (LIPS) integral.

$$\int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} = \int \frac{d^4p}{(2\pi)^4} \delta^2(p^2 - m^2)\, \theta(p^0)$$

Field operator $\phi(\underline{x})$ creates a particle at position $\underline{x}$.

$$\phi(\underline{x})|0\rangle = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} a_p^\dagger |0\rangle e^{-i\underline{p}\cdot\underline{x}}$$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} e^{-i\underline{p}\cdot\underline{x}} |\underline{p}\rangle \sim |\underline{x}\rangle \text{ in NR QM}$$

IN QM, $\langle \underline{p}|\underline{x}\rangle = e^{-i\underline{p}\cdot\underline{x}}$ plane wave state

Here $\langle \underline{p}|\phi(\underline{x})|0\rangle = e^{-i\underline{p}\cdot\underline{x}}$

Heisenberg fields: QM operators are functions of time.
Treats space & time on same footing.

$$\phi(x) = \phi(\underline{x}, t) = e^{iHt}\, \phi(\underline{x})\, e^{-iHt}$$

Heisenberg field — Schrodinger field

Use relations: $[H, a_p^\dagger] = E_p a_p^\dagger \rightarrow H a_p^\dagger = a_p^\dagger (H + E_p)$

$[H, a_p] = -E_p a_p \rightarrow H a_p = a_p (H - E_p)$

Then $e^{iHt} a_p e^{-iHt} = \sum_{n=0}^{\infty} \frac{1}{n!} (iHt)^n a_p e^{-iHt}$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} (it)^n a_p (H - E_p)^n e^{-iHt}$$

$$= a_p e^{i(H-E_p)t} e^{-iHt} = a_p e^{-iE_p t}$$

$$e^{iHt} a_p^\dagger e^{-iHt} = a_p^\dagger e^{iE_p t}$$

$$\phi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left(e^{iHt} a_p e^{-iHt} e^{ip\cdot\underline{x}} + e^{iHt} a_p^\dagger e^{-iHt} e^{-ip\cdot\underline{x}} \right)$$

$$\boxed{\phi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left(a_p e^{-ip\cdot x} + a_p^\dagger e^{ip\cdot x} \right)}$$

$$\pi(x) = e^{iHt}\, \pi(\underline{x})\, e^{-iHt} = \frac{\partial}{\partial t} \phi(x)$$

Klein-Gordon Propagator

$$D(x-y) = \langle 0|\phi(x)\,\phi(y)|0\rangle$$

Amplitude for creating a particle at y, having it propagate to x, and get annihilated at x.

$$D(x-y) = \langle 0| \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \int \frac{d^3q}{(2\pi)^3}\frac{1}{\sqrt{2E_q}} \left(a_p a_q^\dagger\right) e^{-ip\cdot x + iq\cdot y} |0\rangle$$

$$= \int \frac{d^3p}{(2\pi)^3} \int \frac{d^3q}{(2\pi)^3} \frac{1}{\sqrt{4E_pE_q}} (2\pi)^3 \delta^3(p-q)\, e^{-ip\cdot(x-y)}$$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} e^{-ip\cdot(x-y)}$$

The time-ordered propagator (or Feynman propagator) is:

$$D_F(x-y) = \begin{cases} D(x-y) & \text{if } x^0 > y^0 \\ D(y-x) & \text{if } x^0 < y^0 \end{cases}$$

$$= \theta(x^0-y^0)\, D(x-y) + \theta(y^0-x^0)\, D(y-x)$$

$$= \langle 0|\, T\left(\phi(x)\,\phi(y)\right)|0\rangle$$

T = time-ordering symbol: order operators in time from right to left.

A useful way to express D_F is:

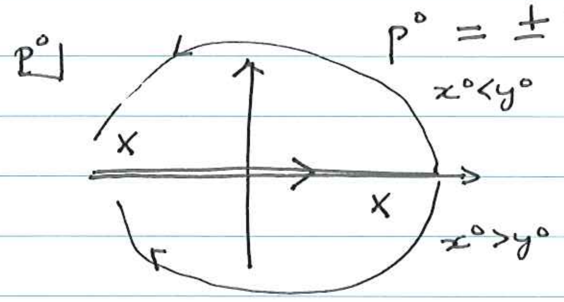
$$\boxed{D_F(x-y) = \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2-m^2+i\varepsilon} e^{-ip\cdot(x-y)}}$$

where ε is a tiny positive number. Note p^0 is not restricted to $p^0 = E_p$, but is integrated over.

Perform $\int_{-\infty}^{\infty} dp^0$ integral using contour integration:

poles at $(p^0)^2 - |\mathbf{p}|^2 - m^2 + i\varepsilon = 0$

$$p^0 = \pm\sqrt{E_p - i\varepsilon} = \pm E_p \mp \frac{i\varepsilon}{2E_p}$$



$$\int_{-\infty}^{\infty} \frac{dp^0}{2\pi} \frac{i}{(p^0-E_p)(p^0+E_p)} e^{-ip^0(x^0-y^0)}$$

$$= \begin{cases} \frac{1}{2\pi}\frac{i}{2E_p}(2\pi i)(-1)\, e^{-iE_p(x^0-y^0)} = \frac{1}{2E_p} e^{-iE_p(x^0-y^0)} & x^0>y^0 \\ \frac{1}{2\pi}\frac{-i}{2E_p}(2\pi i)\, e^{iE_p(x^0-y^0)} = \frac{1}{2E_p} e^{-iE_p(y^0-x^0)} & y^0>x^0 \end{cases}$$

$$D_F(x-y) = \theta(x^0-y^0)\int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_p} e^{-ip\cdot(x-y)}$$

$$+\,\theta(y^0-x^0)\int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_p} e^{-iE_p(y^0-x^0)+i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})}$$

$$= \theta(x^0-y^0)\, D(x-y) + \theta(y^0-x^0)\, D(y-x)$$

Causality

Any two operators $\mathcal{O}_1$ and $\mathcal{O}_2$ must commute if evaluated at spacelike separation.

$$\mathcal{O}_1(x)\,\mathcal{O}_2(y)\,|\text{any state}\rangle = \mathcal{O}_2(y)\,\mathcal{O}_1(x)\,|\text{any state}\rangle$$

measuring $\mathcal{O}_1$ and $\mathcal{O}_2$ in any order gives the same result.
measurements at x & y can't affect each other if $(x-y)^2 < 0$.

$$\Rightarrow \quad [\mathcal{O}_1(x), \mathcal{O}_2(y)] = 0 \quad \text{if} \quad (x-y)^2 < 0 .$$

example: complex field

$$\mathcal{L} = \partial_\mu \phi^* \partial^\mu \phi - m^2 \phi^* \phi$$

$$\phi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_p}} \left(a_p e^{-ip\cdot x} + b_p^\dagger e^{+ip\cdot x} \right)$$

$$\phi^*(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_p}} \left(b_p e^{-ip\cdot x} + a_p^\dagger e^{+ip\cdot x} \right)$$

$a_p^\dagger$ creates particle with charge $+$
$b_p^\dagger$ creates antiparticle with charge $-$

$$\left.\begin{matrix} \\ \\ \end{matrix}\right\} \quad \text{Conserved charge (Noether)} \quad Q = \int \frac{d^3p}{(2\pi)^3} \left(a_p^\dagger a_p - b_p^\dagger b_p \right)$$

$$\mathcal{O}_1(x) = \phi(x) \;, \quad \mathcal{O}_2(y) = \phi^*(y)$$

Evaluate:

$$[\phi(x), \phi^\dagger(y)] = \int \frac{d^3p}{(2\pi)^3} \int \frac{d^3q}{(2\pi)^3} \frac{1}{\sqrt{4\omega_p \omega_q}} \Big([a_p, a_q^\dagger]\, e^{-i(p\cdot x - q\cdot y)}$$
$$+ [b_p^\dagger, b_q]\, e^{+i(p\cdot x - q\cdot y)} \Big)$$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2\omega_p} \left(e^{-ip\cdot(x-y)} - e^{ip\cdot(x-y)} \right)$$

$$= D(x-y) - D(y-x)$$

particle propagating from $y \to x$

antiparticle propagating from $x \to y$

Case: $(x-y)^2 < 0$ (spacelike)

Boost to frame where $(x-y)^\mu = (0, \underline{r})$

$$D(x-y) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2\omega_p} e^{+i\underline{p}\cdot\underline{r}} = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2\omega_p} e^{-i\underline{p}\cdot\underline{r}} = D(y-x)$$

$\Rightarrow$ $D(x-y) - D(y-x) = 0$ in any frame where $(x-y)^2 < 0$.

case: $(x-y)^2 > 0$ (timelike) $\rightarrow$ boost to frame where $(x-y)^\mu = (t,0,0,0)$

$$D(x-y) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2\omega_p} e^{-i\omega_p t} \neq D(y-x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2\omega_p} e^{+i\omega_p t}$$

Causality ensured: amplitude for spacelike propagation of particles cancelled by spacelike propagation of antiparticles. Antiparticles required by causality (particle can be its own antiparticle)