IV. Interacting fields and perturbation theory (Ch 4, P&S)

Up to now, studied free (scalar) field theory (non-interacting particles).

Simplest interacting field theory: ϕ^4-theory

$$\mathcal{L} = \underbrace{\tfrac{1}{2}(\partial_\mu \phi)^2 - \tfrac{1}{2}m^2\phi^2}_{\mathcal{L}_0} \underbrace{- \tfrac{1}{4!}\lambda\phi^4}_{\mathcal{L}_{int}}$$

where λ is a coupling constant.

$\mathcal{L}_0$ = free Lagrangian → can be solved exactly

$\mathcal{L}_{int}$ = interaction Lagrangian

dimensional analysis:

S is dimensionless (same units as $\hbar$)

$S = \int d^4x\, \mathcal{L}$ → $\mathcal{L}$ has mass dim 4

∂_μ has mass dim 1

ϕ has mass dim 1

m has mass dim 1

λ has mass dim 0 (dimensionless coupling constant)

Effect of $\mathcal{L}_{int}$ treated using perturbation theory

Example: time-ordered propagator (two point correlation function)

$$\langle \Omega | T(\phi(x)\phi(y)) | \Omega \rangle = D_F(x-y) + \mathcal{O}(\lambda) + \mathcal{O}(\lambda^2) + \dots$$

$|\Omega\rangle$ = ground state (vacuum) of full interacting theory $\neq |0\rangle$ ground state of free theory

$\phi(x)$ = Heisenberg field of full interacting theory.

$$= e^{iH(t-t_0)} \underbrace{\phi(t_0,\underline{x})}_{\text{Schrodinger picture field}} e^{-iH(t-t_0)}$$

Interaction picture:

$$\phi_I(t,\underline{x}) = e^{iH_0(t-t_0)} \phi(t_0,\underline{x})\, e^{-iH_0(t-t_0)}$$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left(a_{\underline{p}} e^{-ip\cdot x} + a_{\underline{p}}^\dagger e^{ip\cdot x}\right) \qquad x^0 = t-t_0$$

Same as Heisenberg field in free theory.

Full interacting Heisenberg field is:

$$\phi(t,\underline{x}) = e^{iH(t-t_0)} e^{-iH_0(t-t_0)} \phi_I(t,\underline{x})\, e^{iH_0(t-t_0)} e^{-iH(t-t_0)}$$

$$= U^\dagger(t,t_0)\, \phi_I(t,\underline{x})\, U(t,t_0)$$

where $U(t,t_0) = e^{iH_0(t-t_0)} e^{-iH(t-t_0)}$ is a unitary operator: time-evolution operator.

Note: $$\frac{\partial}{\partial t} U(t,t_0) = i\, e^{iH_0(t-t_0)} (H_0 - H)\, e^{-iH(t-t_0)}$$

$$= -i\, e^{iH_0(t-t_0)} H_{int}\, e^{-iH_0(t-t_0)}\, U(t,t_0)$$

$$= -i\, H_I\, U(t,t_0)$$

where $H_I = e^{iH_0(t-t_0)} H_{int}\, e^{-iH_0(t-t_0)}$ is H_{int} in interaction picture here: $H_I = \int d^3x \, \frac{\lambda}{4!} \, e^{iH_0(t-t_0)} \, \phi(t_0, \underline{x})^4 \, e^{-iH_0(t-t_0)}$

$= \int d^3x \, \frac{\lambda}{4!} \, \phi_I^4$

Solution for $U(t, t_0)$:

$$U(t, t_0) = 1 + (-i)\int_{t_0}^{t} dt_1 \, H_I(t_1) + (-i)^2 \int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2 \, H_I(t_1) H_I(t_2)$$
$$+ (-i)^3 \int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2 \int_{t_0}^{t_2} dt_3 \, H_I(t_1) H_I(t_2) H_I(t_3) + \dots$$

$$\frac{\partial}{\partial t} U(t, t_0) = 0 + (-i) H_I(t) + (-i)^2 \int_{t_0}^{t} dt_2 \, H_I(t) H_I(t_2)$$
$$+ (-i)^3 \int_{t_0}^{t} dt_2 \int_{t_0}^{t_2} dt_3 \, H_I(t) H_I(t_2) H_I(t_3) + \dots$$

$$= (-i) H_I(t) \, U(t, t_0)$$

Note: $\int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2 \, H_I(t_1) H_I(t_2)$

$$= \frac{1}{2} \int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2 \, H_I(t_1) H_I(t_2) \qquad t_1 > t_2$$
$$+ \frac{1}{2} \int_{t_0}^{t} dt_2 \int_{t_0}^{t_2} dt_1 \, H_I(t_2) H_I(t_1) \qquad t_2 > t_1$$

$$= \frac{1}{2} \int_{t_0}^{t} dt_1 \int_{t_0}^{t} dt_2 \, H_I(t_1) H_I(t_2) \, \theta(t_1 - t_2)$$
$$+ \frac{1}{2} \int_{t_0}^{t} dt_1 \int_{t_0}^{t} dt_2 \, H_I(t_2) H_I(t_1) \, \theta(t_2 - t_1)$$

$$= \frac{1}{2} \int_{t_0}^{t} dt_1 \int_{t_0}^{t} dt_2 \, T\left(H_I(t_1) H_I(t_2) \right)$$

Generalizes to higher order terms:

$$U(t,t_0) = 1 + (-i)\int_{t_0}^{t} dt_1\, H_I(t_1) + \frac{1}{2!}(-i)^2 \int_{t_0}^{t} dt_1 \int_{t_0}^{t} dt_2\, T\big(H_I(t_1) H_I(t_2)\big)$$

$$+ \frac{1}{3!}(-i)^3 \int_{t_0}^{t} dt_1 \int_{t_0}^{t} dt_2 \int_{t_0}^{t} dt_3\, T\big(H_I(t_1) H_I(t_2) H_I(t_3)\big) + \cdots$$

$$= T\left(\exp\left[-i\int_{t_0}^{t} dt'\, H_I(t')\right]\right)$$

More generally:

$$U(t_1, t_2) = T\left(\exp\left[-i\int_{t_1}^{t_2} dt\, H_I(t)\right]\right)$$

$$= U(t_1, t_0)\, U(t_2, t_0)^{\dagger}$$

$$= e^{iH_0(t_1-t_0)} e^{-iH(t_1-t_0)} e^{iH(t_2-t_0)} e^{-iH_0(t_2-t_0)}$$

$$= e^{iH_0(t_1-t_0)} e^{-iH(t_1-t_2)} e^{-iH_0(t_2-t_0)}$$

Time evolution operator is unitary; satisfies

$$U(t_1, t_2)\, U(t_2, t_3) = U(t_1, t_3)$$

$$U(t_1, t_2) = U(t_2, t_1)^{\dagger}$$

Evaluate interacting vacuum $|\Omega\rangle$ in terms of $|0\rangle$

expand: $|0\rangle = \sum_n |n\rangle\langle n|0\rangle = |\Omega\rangle\langle\Omega|0\rangle + \sum_{n\neq\Omega} |n\rangle\langle n|0\rangle$

where $|n\rangle$ are eigenstates of H.

Act on $|0\rangle$ with $e^{-iH(t_0+T)}$:

$$e^{-iH(t_0+T)}|0\rangle = e^{-iE_0(t_0+T)}|\Omega\rangle\langle\Omega|0\rangle + \sum_{n\neq\Omega} e^{-iE_n(t_0+T)}|n\rangle\langle n|0\rangle$$

where $H|\Omega\rangle = E_0|\Omega\rangle$
$H|n\rangle = E_n|n\rangle$ $\quad (E_n > E_\Omega)$

note: $H_0|0\rangle = 0$ but $H|\Omega\rangle \neq 0$.

Take limit $T \to \infty \times (1-i\varepsilon)$

$\cancel{e^{-iH}}$ $e^{-iE_0 T} \to 0$ slower than $e^{-iE_n T} \to 0$

$$\lim_{T\to\infty(1-i\varepsilon)} e^{-iH(t_0+T)}|0\rangle = \lim e^{-iE_0(t_0+T)}|\Omega\rangle\langle\Omega|0\rangle$$

$$\Rightarrow \quad |\Omega\rangle = \lim_{T\to\infty(1-i\varepsilon)} \frac{e^{-iH(t_0+T)}}{e^{-iE_0(t_0+T)}\langle\Omega|0\rangle}|0\rangle$$

$$= \lim_{T\to\infty(1-i\varepsilon)} \frac{e^{-iH(t_0+T)}e^{iH_0(t_0+T)}}{e^{-iE_0(t_0+T)}\langle\Omega|0\rangle}|0\rangle$$

$$= \lim_{T\to\infty(1-i\varepsilon)} \frac{U^{\dagger}(-T,t_0)\,|0\rangle}{e^{-iE_0(t_0+T)}\langle\Omega|0\rangle}$$

$$= \lim_{T\to\infty(1-i\varepsilon)} \frac{U(t_0,-T)\,|0\rangle}{e^{-iE_0(t_0+T)}\langle\Omega|0\rangle}$$

Similar expression for $\langle\Omega|$ by

$$\langle 0|\, e^{iH(T-t_0)} = \langle 0|\Omega\rangle\langle\Omega|\, e^{iE_0(T-t_0)} + \sum_n \cdots$$

$$\Rightarrow\ \langle\Omega| = \lim_{T\to\infty(1-i\varepsilon)} \langle 0|\,U(T,t_0)\,\frac{1}{e^{-iE_0(T-t_0)}\langle 0|\Omega\rangle}$$

Now evaluate propagator: $\langle\Omega|T(\phi(x)\phi(y))|\Omega\rangle$

Take $x^0 > y^0$:

$$\langle\Omega|\phi(x)\,\phi(y)|\Omega\rangle = \frac{\langle\Omega|\phi(x)\,\phi(y)|\Omega\rangle}{\langle\Omega|\Omega\rangle} \quad \text{since } \langle\Omega|\Omega\rangle = 1$$

$$= \lim_{T\to\infty(1-i\varepsilon)} \frac{\langle 0|U(T,t_0)\,U^{\dagger}(x_0,t_0)\,\phi_I(x)\,U(x_0,t_0)\,U^{\dagger}(y^0,t_0)\,\phi_I(y)\,U(y^0,t_0) \times U(t_0,-T)\,|0\rangle}{\langle 0|\,U(T,t_0)\,U(t_0,-T)\,|0\rangle}$$

$$= \lim_{T\to\infty(1-i\varepsilon)} \frac{\langle 0|\,U(T,x_0)\,\phi_I(x)\,U(x_0,y_0)\,\phi_I(y)\,U(y^0,-T)|0\rangle}{\langle 0|\underbrace{U(T,t_0)\,U(t_0,-T)}_{U(T,-T)}|0\rangle}$$

$$= \lim_{T\to\infty(1-i\varepsilon)} \frac{\langle 0|\,T\left(\phi_I(x)\,\phi_I(y)\,\exp\left[-i\int_{-T}^{T}dt\ H_I(t)\right]\right)|0\rangle}{\langle 0|\,T\left(\exp\left[-i\int_{-T}^{T}dt\ H_I(t)\right]\right)|0\rangle}$$

<u>Perturbation theory:</u> treat interaction coupling constant λ as a small parameter to be expanded in.

$$\exp\left(-i\int dt\, H_I(t)\right) = \exp\left(i\int d^4x\, \mathcal{L}_I(x)\right)$$

where $\mathcal{L}_I = \frac{\lambda}{4!}\phi_I^4$

So expanding

$$\exp\left(i\int d^4x\, \mathcal{L}_I\right) = \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{i\lambda}{4!}\int d^4x\, \phi_I^4(x)\right)^n$$ amounts

to an expansion in powers of λ^n.

At a given order in λ, need to compute a time-ordered product of

$$\langle 0 | T\left(\phi_I(x_1)\phi_I(x_2)\cdots\phi(x_n)\right)|0\rangle .$$

e.g.

$$\langle 0 | T\left(\phi_I(x)\phi_I(y)\exp\left[-i\int dt\, H_I(t)\right]\right)|0\rangle$$

$$= \underbrace{\langle 0 | T(\phi_I(x)\phi_I(y))|0\rangle}_{\text{free propagator we already computed.}} + i\frac{\lambda}{4!}\int d^4z\, \langle 0 | T\phi_I(x)\phi_I(y)\phi_I^4(z)|0\rangle + \cdots$$

Wick's theorem

Want to evaluate $\langle 0 | T(\phi_I(x_1)\phi_I(x_2)\cdots\phi_I(x_n)) | 0\rangle$

Normal ordering

Recall: $\phi_I(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left(a_p e^{-ip\cdot x} + a_p^\dagger e^{ip\cdot x} \right)$

Useful to write: $\phi_I(x) = \phi_I^+(x) + \phi_I^-(x)$

where
$$\phi_I^+(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} a_p e^{-ip\cdot x}$$
$$\phi_I^-(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} a_p^\dagger e^{ip\cdot x}$$

Note: $\phi_I^+(x)|0\rangle = 0$ and $\langle 0|\phi_I^-(x) = 0$.

Normalordering means all a_p's are to the right of $a_p^\dagger$'s
(or ϕ_I^+ fields to the right of ϕ_I^- fields)

$$N(a_p a_k^\dagger a_q) = a_k^\dagger a_p a_q$$

$$N(\phi_I(x)\phi_I(y)) = N\left((\phi_I^+(x) + \phi_I^-(x))(\phi_I^+(y) + \phi_I^-(y))\right)$$
$$= \phi_I^+(x)\phi_I^+(y) + \phi_I^-(y)\phi_I^+(x) + \phi_I^-(x)\phi_I^+(y) + \phi_I^-(x)\phi_I^-(y)$$

evaluate $T(\phi_I(x)\,\phi_I(y))$ (note: drop I $\rightarrow$ always in interaction picture)

$$T(\phi(x)\,\phi(y)) \underset{\substack{\uparrow \\ \text{case } x^0 > y^0}}{=} \phi(x)\,\phi(y)$$

$$= \phi^+(x)\,\phi^+(y) + \phi^-(x)\,\phi^+(y) + \phi^+(x)\,\phi^-(y) + \phi^-(x)\,\phi^-(y)$$

$$= N(\phi(x)\phi(y)) + [\phi^+(x), \phi^-(y)]$$

$$T(\phi(x)\,\phi(y)) = N(\phi(x)\phi(y)) + \begin{cases} [\phi^+(x), \phi^-(y)] & x^0 > y^0 \\ [\phi^+(y), \phi^-(x)] & y^0 > x^0 \end{cases}$$

$$[\phi^+(x), \phi^-(y)] = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}} \int \frac{d^3q}{(2\pi)^3}\frac{1}{\sqrt{2E_q}}\, e^{-i(p\cdot x - q\cdot y)} \underbrace{[a_p, a_q^\dagger]}_{(2\pi)^3\delta^3(p-q)}$$

$$= \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_p}\, e^{-ip\cdot(x-y)} = D(x-y)$$

$$[\phi^+(y), \phi^-(x)] = D(y-x)$$

$$T(\phi(x)\,\phi(y)) = N(\phi(x)\,\phi(y)) + D_F(x-y)$$

$$\langle 0|T(\phi(x)\,\phi(y))|0\rangle = D_F(x-y) \qquad \text{since } \langle 0|N(\ldots)|0\rangle = 0.$$

Write this as $T(\phi(x)\,\phi(y)) = N(\phi(x)\,\phi(y)) + \overbracket{\phi(x)\,\phi(y)}$

$$\overbracket{\phi(x)\,\phi(y)} = D_F(x-y)$$ <u>contraction</u> of two fields

Since D_F is a number: $T(\phi(x)\phi(y)) = N(\phi(x)\phi(y) + \overbracket{\phi(x)\,\phi(y)})$

Wick's theorem:

$$T(\text{ product of fields }) = N(\text{all possible contractions}) \quad \text{(product of fields + all possible contractions)}$$

$$T(\phi(x_1)\phi(x_2)) = N(\phi_1\phi_2 + \phi_1\phi_2)$$

$$T(\phi_1\phi_2\phi_3) = N(\phi_1\phi_2\phi_3 + \phi_1\phi_2\phi_3 + \phi_1\phi_2\phi_3 + \phi_1\phi_2\phi_3)$$

$$= N(\phi_1\phi_2\phi_3) + D_F(x_1 - x_2)\, N(\phi_3)$$

$$+ D_F(x_1 - x_3)\, N(\phi_2) + D_F(x_1 - x_3)\, N(\phi_3)$$

Want to compute: $\langle 0 | T(\phi_1\phi_2 \ldots) | 0 \rangle$
vacuum expectation value of time-ordered product

note: $\langle 0 | N(\phi_1\phi_2 \ldots) | 0 \rangle = 0$
only terms where all fields are contracted survive.

$$\langle 0 | T(\phi_1\phi_2) | 0 \rangle = D_F(x_1 - x_2)$$

$$\langle 0 | T(\phi_1\phi_2\phi_3\phi_4) | 0 \rangle$$

$$= \langle 0 | N(\phi_1\phi_2\phi_3\phi_4 + \phi_1\phi_2\phi_3\phi_4 + \phi_1\phi_2\phi_3\phi_4 + \ldots) | 0 \rangle$$

$$= D_F(x_1 - x_2)\, D_F(x_3 - x_4) + D_F(x_1 - x_3)\, D_F(x_2 - x_4)$$

$$+ D_F(x_1 - x_4)\, D_F(x_2 - x_3)$$

Proof of Wick's theorem: proof by induction
we proved it for 2 fields.
Assume true for n fields:

$$T(\phi_1 \cdots \phi_n) = N(\phi_1 \cdots \phi_n + \text{all possible contractions})$$

Consider $n+1$ fields (with $x^0_{n+1} < x^0_1, .., x^0_n$)

$$T(\phi_1 \cdots \phi_{n+1}) = T(\phi_1 \cdots \phi_n)\,\phi_{n+1}$$

$$= N(\phi_1 \cdots \phi_n + \text{all possible contractions})\,(\phi^+_{n+1} + \phi^-_{n+1})$$

$$= N(\phi_1 \cdots \phi_{n+1} + \text{all possible contractions not involving } \phi_{n+1})$$

$$+ \left[N(\phi_1 \cdots \phi_n + \text{all contractions}),\ \phi^-_{n+1} \right]$$

evaluate 2nd term:

$$\left[N(\phi_1 \cdots \phi_n),\ \phi^-_{n+1} \right] = N\Big([\phi^+_1, \phi^-_{n+1}]\,\phi_2 \cdots \phi_n$$
$$+ \phi_1 [\phi^+_2, \phi^-_{n+1}]\,\phi_3 \cdots \phi_n + \cdots$$
$$+ \phi_1 \cdots \phi_{n-1} [\phi^+_n, \phi^-_{n+1}] \Big)$$

$$= N\Big(\contraction{}{\phi_1}{\phi_2 \cdots \phi_n}{\phi_{n+1}} \phi_1 \phi_2 \cdots \phi_n \phi_{n+1} + \contraction{\phi_1}{\phi_2}{\cdots \phi_n}{\phi_{n+1}} \phi_1 \phi_2 \cdots \phi_n \phi_{n+1}$$
$$+ \cdots + \contraction{\phi_1 \phi_2 \cdots}{\phi_n}{}{\phi_{n+1}} \phi_1 \phi_2 \cdots \phi_n \phi_{n+1} \Big)$$

$$= N(\text{all possible single contractions with } \phi_{n+1})$$

$$\left[N(\text{single contractions of } \phi_1 \cdots \phi_n),\ \phi^-_{n+1} \right] = N(\text{all possible double contractions with } \phi_{n+1})$$

So 2nd term = all possible contractions involving ϕ_{n+1}

proved for any number of fields.

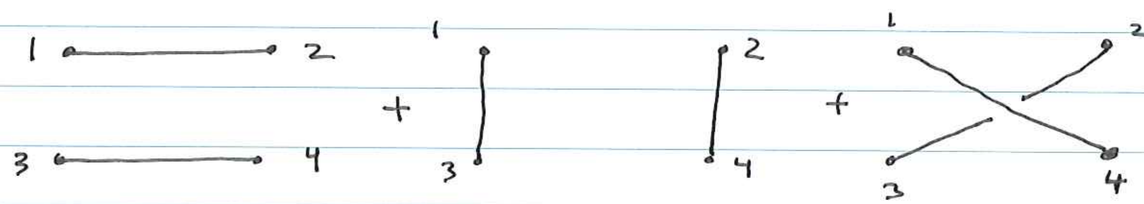
Feynman diagrams

$D_F(x-y)$ can be thought of as a particle propagating between x and y.

Represented diagramatically by 

$$\langle 0 | T(\phi_I(x_1)\phi_I(x_2)\phi_I(x_3)\phi_I(x_4)) | 0 \rangle$$

$$= D_F(x_1-x_2)D_F(x_3-x_4) + D_F(x_1-x_3)D_F(x_2-x_4)$$
$$+ D_F(x_1-x_4)D_F(x_2-x_3)$$

= 

Now evaluate: $\langle \Omega | T(\phi(x)\phi(y)) | \Omega \rangle$

$$= \text{const} \times \langle 0 | T(\phi_I(x)\phi_I(y) \exp[+i\int d^4z \mathcal{L}_I(z)]) | 0 \rangle$$

$$= \text{const} \times \Big(\langle 0 | T(\phi_I(x)\phi_I(y)) | 0 \rangle$$
$$+ \frac{-i\lambda}{4!}\int d^4z \langle 0 | T(\phi(x)\phi(y)\phi(z)^4) | 0 \rangle + \cdots \Big)$$

$$\langle 0 | T(\phi(x)\phi(y)\phi(z)\phi(z)\phi(z)\phi(z)) | 0 \rangle$$

$$= \langle 0 | \phi_x \phi_y \phi_z \phi_z \phi_z \phi_z \times 3$$

$$+ \phi_x \phi_y \phi_z \phi_z \phi_z \phi_z \times 4 \times 3 \; | 0 \rangle$$

$$= 3 D_F(x-y)\, D_F(z-z)^2 + 12 D_F(x-z) D_F(y-z) D_F(z-z)$$

$= 3$  $+ \; 12$ 

$$\langle \Omega | T(\phi(x)\phi(y)) | \Omega \rangle$$

$$= \text{const} \times \Big(D_F(x-y) - \frac{i}{8}\lambda \int d^4z\, D_F(z-z)^2 D_F(x-y)$$

$$+ \frac{-i}{2}\lambda \int d^4z\, D_F(x-z) D_F(y-z) D_F(z-z) + \cdots \Big)$$

Feynman diagrams give a useful prescription for organizing and computing amplitudes.

x, y = external points ; z = vertex.

order λ^n term in two point correlation function is:

$$\langle 0 | T(\phi(x)\phi(y) \frac{1}{n!} (i \int d^4z_1 \mathcal{L}_I(z_1)) \times \cdots \times (i \int d^4z_n \mathcal{L}_I(z_n))) | 0 \rangle$$

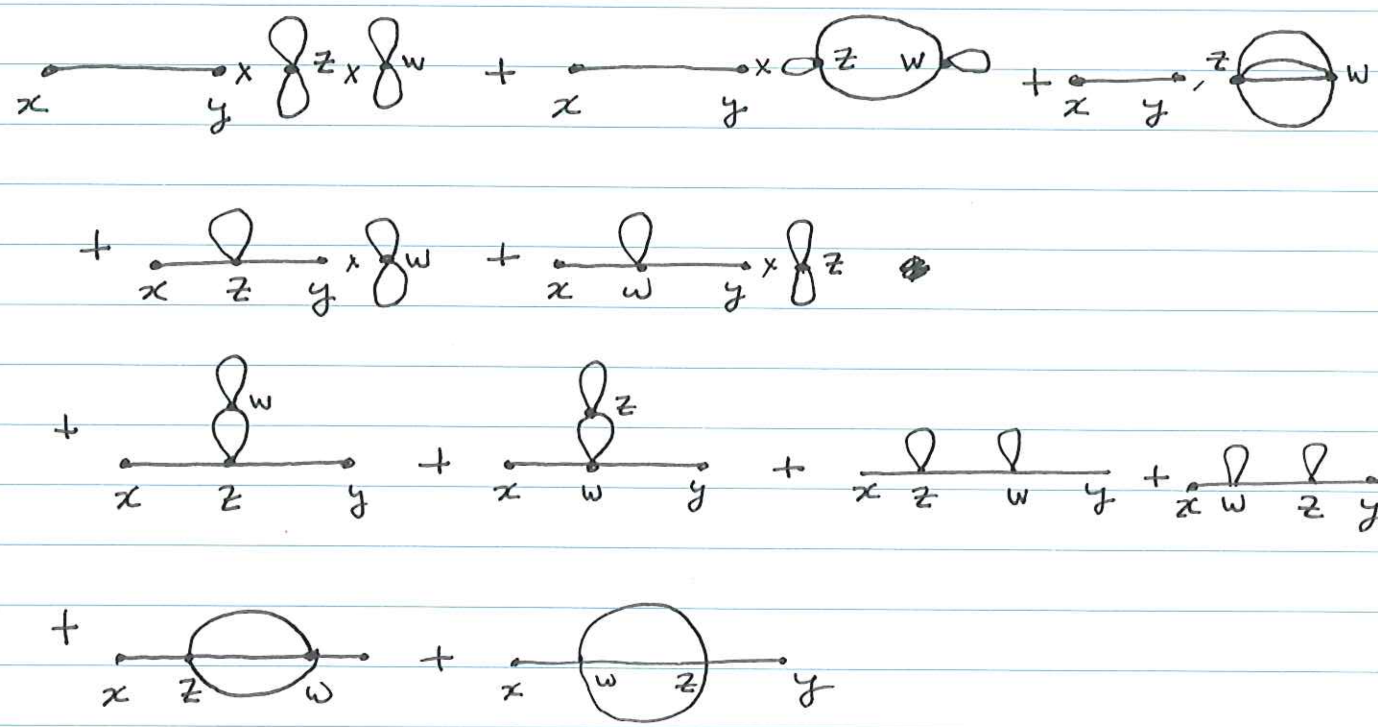
= all possible diagrams with two external points and n vertices.

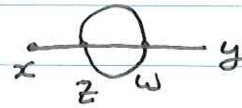
(1) Draw all possible diagrams.

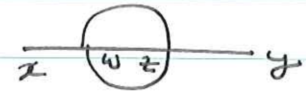
(2) Compute each diagram using Feynman rules

- each propagator x •——• y $= D_F(x-y)$
- each vertex $\times z$ $= \frac{(-i\lambda)}{4!} \int d^4z$
- each external point x •—— $= 1$
- multiply by symmetry factor $= \frac{1}{n!} \times$ number of possible contractions

Example: two-point correlation function at $\mathcal{O}(\lambda^2)$:



example: sunset diagrams 

+ 

$$= \left(\frac{-i\lambda}{4!}\right)^2 \frac{1}{2!} \int d^4z \int d^4w \left[D_F(x-z)\, D_F(z-w)^3\, D_F(w-y) \right.$$
$$\left. + D_F(x-w)\, D_F(z-w)^3\, D_F(z-y) \right] \times 4 \cdot 4!$$

number of contractions:

$$\phi_x\ \phi_y\ \phi_z\ \phi_z\ \phi_z\ \phi_z\ \phi_w\ \phi_w\ \phi_w\ \phi_w$$

4 ways to contract x & z
4 ways to contract y & w
3 ways to contract 1st z & w $\times$ 2 ways for 2nd $\times$ 1 way for 3rd

$$= 4 \times 4 \times 3! = 4 \cdot 4!$$

$$= \frac{-\lambda^2}{4!\,\cancel{4!}} \frac{1}{\cancel{2}} \cancel{4!}\, 4 \cdot \cancel{2} \int d^4z \int d^4w\, D_F(x-z)\, D_F(z-w)^3\, D_F(w-y)$$

$$= -\frac{\lambda^2}{6} \int d^4z \int d^4w\, D_F(x-z)\, D_F(z-w)^3\, D_F(w-y)$$

note: exchanging $z \leftrightarrow w$ picks up a factor of 2 which cancels $\frac{1}{2!}$ from exponential.
typically, exchanging internal points gets a factor of $n!$ which cancels $\frac{1}{n!}$ from exponential.

Position Space Feynman rules:

- propagator 

$= D_F(x-y)$

- vertex 

$= (-i\lambda) \int d^4z$

- external point 

$= 1$

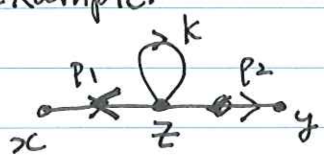
- multiply by symmetry factor

$$\frac{1}{n!}\left(\frac{1}{4!}\right)^n \times \underbrace{n!}_{\text{exchange of vertices}} \times \ \# \text{ possible contractions.}$$

Can also write a diagram in momentum space:

$$D_F(x-y) = \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2-m^2+i\varepsilon} e^{-ip\cdot(x-y)}$$

example:



$$= \frac{(-i\lambda)}{2} \int d^4z \, D_F(x-z) D_F(y-z) D_F(z-z)$$

$$= \frac{-i\lambda}{2} \int d^4z \int \frac{d^4p_1}{(2\pi)^4} \frac{i}{p_1^2-m^2+i\varepsilon} e^{-ip_1\cdot(x-z)} \int \frac{d^4p_2}{(2\pi)^4} \frac{i}{p_2^2-m^2+i\varepsilon} e^{-ip_2\cdot(y-z)}$$

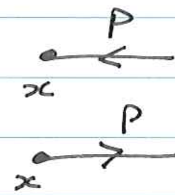
$$\times \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2-m^2+i\varepsilon}$$

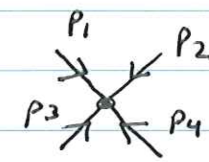
$$= \frac{-i\lambda}{2} \int \frac{d^4p_1}{(2\pi)^4} \int \frac{d^4p_2}{(2\pi)^4} \int \frac{d^4k}{(2\pi)^4} e^{-ip_1\cdot x} e^{-ip_2\cdot y} (2\pi)^4 \delta^4(p_1+p_2)$$

Momentum Space Feynman rules: Assign a momentum to each propagator, where arrow denotes direction of momentum.

- propagator  $= \dfrac{i}{p^2 - m^2 + i\varepsilon}$

- vertex  $= -i\lambda$

- external point  $= e^{-ip\cdot x}$, $= e^{ip\cdot x}$

- impose momentum conservation at each vertex 
$(2\pi)^4 \delta^4(p_1+p_2+p_3+p_4)$

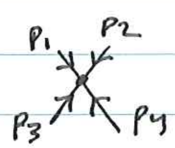
- integrate over all momenta $\int \frac{d^4p}{(2\pi)^4}$

- multiply by symmetry factor.

Lastly, remember

$$\int d^4z = \lim_{T\to\infty(1-i\varepsilon)} \int_{-T}^{T} dz^0 \int d^3z$$

So consider a vertex



$$\int d^4z \, e^{-i(p_1+p_2+p_3+p_4)\cdot z} = \lim_{T\to\infty(1-i\varepsilon)} \int_{-T}^{T} dz^0 \, e^{-i(p_1^0+p_2^0+p_3^0+p_4^0)z^0} \times (2\pi)^3 \delta^3(p_1+p_2+p_3+p_4)$$

the shift $(1-i\varepsilon)$ can be absorbed by defining

$$z^0 = (1-i\varepsilon)z'^0 \qquad p_i^0 = (1+i\varepsilon)p_i'^0$$

$$= \int_{-\infty}^{\infty} dz'^0 \, e^{-i(p_1'^0+p_2'^0+p_3'^0+p_4'^0)z'^0} (2\pi)^3\delta^3(..) = (2\pi)^4\delta^4(p_1+p_2+p_3+p_4)$$

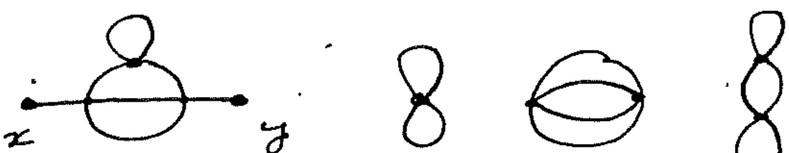
but then $p^2 \to p'^2 + i\varepsilon$ $\to$ equivalent to our choice of $+i\varepsilon$ in Feynman prop.

Vacuum Bubbles:

$$\langle 0| T\left(\phi_I(x)\, \phi_I(y)\, \exp\left[i \int d^4z\, \mathcal{L}_I(z)\right]\right)|0\rangle$$

$= \sum$ all possible diagrams with external points x and y.

A given diagram is a product of connected and disconnected subdiagrams.

e.g.: 

$\sim \mathcal{O}(\lambda^8)$

connected (to external points)

disconnected vacuum bubbles

Exponentiation of vacuum bubbles:

Consider lowest order vacuum bubble: 

Let's pick out all terms in $\langle 0|T(\phi_I(x)\phi_I(y)\exp[\cdot\cdot])|0\rangle$ with this diagram.

Note:  $= \int d^4z \frac{-i\lambda}{4!}\, \phi_I(z)\, \phi_I(z)\, \phi_I(z)\, \phi_I(z) \times \left(\begin{matrix}3 \text{ possible}\\ \text{contractions}\end{matrix}\right)$

Consider nth term in Taylor expansion:

$$\frac{1}{n!}\langle 0|\, T\, \phi_I(x)\, \phi_I(y)\, i\int d^4z_1\, \mathcal{L}_I(z_1) \cdots i\int d^4z_n\, \mathcal{L}_I(z_n)\, |0\rangle$$

This term can have 0, 1, .., n bubbles like 8

The term with m bubbles is:

$$\frac{1}{n!}\langle 0|\, T\, \phi_I(x)\, \phi_I(y)\, i\int d^4z_1\, \mathcal{L}_I(z_1) \cdots i\int d^4z_{n-m}\, \mathcal{L}_I(z_{n-m})\, |0\rangle\Big|_{\substack{\text{no } 8 \\ \text{bubbles}}}$$

$$\times \binom{n}{m} \left(8\right)^m$$

$$\binom{n}{m} = \frac{n!}{(n-m)!\,m!} = $$ # ways of picking m different choices given n possibilities.

The full Taylor expansion is:

$$\sum_{n=0}^{\infty} \frac{1}{n!}\langle 0|\, T\, \phi_I(x)\, \phi_I(y)\, i\int d^4z_1\, \mathcal{L}_I(z_1) \cdots i\int d^4z_n\, \mathcal{L}_I(z_n)\, |0\rangle$$

$$= \sum_{m=0}^{\infty} \sum_{n=m}^{\infty} \frac{1}{n!}\langle 0|\, T\, \phi_I(x)\, \phi_I(y)\, i\int d^4z_1\, \mathcal{L}_I(z_1) \cdots i\int d^4z_{n-m}\, \mathcal{L}_I(z_{n-m})\, |0\rangle\Big|_{\text{no } 8}$$

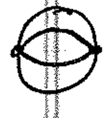
$$\times \frac{n!}{(n-m)!\,m!}\; 8^m$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!}\langle 0|\, T\, \phi_I(x)\, \phi_I(y)\, i\int d^4z_1\, \mathcal{L}_I(z_1) \cdots i\int d^4z_n\, \mathcal{L}_I(z_n)\, |0\rangle\Big|_{\text{no } 8}$$

$$\times \sum_{m=0}^{\infty} \frac{1}{m!}\, 8^m$$

$$= \langle 0|\, T\, \phi_I(x)\, \phi_I(y)\, \exp\!\left[i\int d^4z\, \mathcal{L}_I(z)\right] |0\rangle\Big|_{\text{no } 8}\; e^{8}$$

Consider another vacuum bubble:

z  w $$= \int d^4z \int d^4w \left(\frac{-i\lambda}{4!}\right)^2 \phi_I(z)\phi_I(z)\phi_I(z)\phi_I(z)\phi_I(w)\phi_I(w)\phi_I(w)\phi_I(w)$$

$\times$ (4! possible contractions)

consider ~~term~~ nth order term with m bubbles:

$$\frac{1}{n!}\langle 0|T\phi_I(x)\phi_I(y)\, i\int d^4z_1 \mathcal{L}_I(z_1)\cdots i\int d^4z_{n-2m}\mathcal{L}_I(z_{n-2m})|0\rangle\Big|_{\text{no}}$$

$\times$  ^{m} $\times$ combinatoric factor

$$= \binom{n}{2}\binom{n-2}{2}\cdots\binom{n-2m+2}{2}\times\frac{1}{m!}$$

$$= \frac{1}{2^m}\frac{1}{m!}\frac{n!}{(n-2m)!}$$

$$\sum_{n=0}^{\infty}\frac{1}{n!}\langle 0|T\phi_I(x)\phi_I(y)\exp[\cdots]|0\rangle$$

$$= \sum_{m=0}^{\infty}\sum_{n=2m}^{\infty}\frac{1}{2^m m!}\frac{1}{(n-2m)!}\langle 0|T\phi_I(x)\phi_I(y)\, i\int d^4z_1\mathcal{L}_I(z_1)\cdots$$

$$\cdots i\int d^4z_{n-2m}\mathcal{L}_I(z_{n-2m})|0\rangle\Big|_{\text{no}}$$

$\times$  ^{m}

$$= \langle 0|T\phi_I(x)\phi_I(y)\exp\left[i\int d^4z\,\mathcal{L}_I(z)\right]|0\rangle\Big|_{\text{no}} \times e^{\frac{1}{2}\,\ominus}$$

General vacuum bubble B_i with k vertices

(e.g. $B_0 =$  , 1 vertex

$B_1 =$  , 2 vertices ....)

Consider $\mathcal{O}(\lambda^n)$ term with m bubbles B_i :

Each B_i involves contracting k factors of $\int d^4z \mathcal{L}_I(z)$

$$\frac{1}{n!}\langle 0 | T \phi_I(x)\phi_I(y)\, i\int d^4z_1 \mathcal{L}_I(z_1) \cdots i\int d^4 z_{n-mk} \mathcal{L}_I(z_{n-mk}) |0\rangle \Big|_{\text{no } B_i}$$

$$\times \quad (B_i)^m \times \text{combinatoric factor}$$

$$\downarrow$$

$$= \binom{n}{k}\binom{n-k}{k}\cdots\binom{n-k(m-1)}{k}\cdot\frac{1}{m!}$$

$$= \frac{n!}{k!(n-k)!}\,\frac{(n-k)!}{k!(n-2k)!}\cdots\frac{(n-k(m-1))!}{k!\,(n-mk)!}\,\frac{1}{m!}$$

$$= \frac{n!}{(k!)^m(n-mk)!}\,\frac{1}{m!}$$

$$\langle 0 | T \phi_I(x)\phi_I(y) \exp\left[i\int d^4z\, \mathcal{L}_I(z)\right]|0\rangle \Big|_{\text{terms with } B_i}$$

$$= \sum_{m=0}^{\infty}\sum_{n=mk}^{\infty} \frac{1}{(n-mk)!}\langle 0|T\phi_I(x)\phi_I(y) \left(i\int d^4z\, \mathcal{L}_I(z)\right)^{n-mk}|0\rangle_{\text{no } B_i}$$

$$\times \frac{1}{(k!)^m}\,\frac{1}{m!}\,(B_i)^m$$

$$= \langle 0|T\, \phi_I(x)\, \phi_I(y) \exp\left[i\int d^4z\, \mathcal{L}_I(z)\right]|0\rangle_{\text{no } B_i}$$

$$\times \quad \exp\left(\frac{1}{k!}B_i\right)$$

All vacuum bubbles can be factored out of the two-pt. function.

$$\langle 0|T\,\phi_I(x)\,\phi_I(y)\,\exp\left[i\int d^4z\,\mathcal{L}_I(z)\right]|0\rangle$$

$$= \langle 0|T\,\phi_I(x)\phi_I(y)\exp\left[i\int d^4z\,\mathcal{L}_I(z)\right]|0\rangle\Big|_{\text{no vacuum bubbles}}$$

$$\times \exp\left(\textstyle\sum_i B_i\right)$$

Note: same argument applies to

$$\langle 0|T\,\exp\left[i\int d^4z\,\mathcal{L}_I(z)\right]|0\rangle = \langle 0|0\rangle\cdot\exp\left(\textstyle\sum_i B_i\right)$$

$$= \exp\left(\textstyle\sum_i B_i\right)$$

no external points $\rightarrow$ only vacuum bubbles contribute.

Full two-point function:

$$\langle\Omega|T\,\phi(x)\,\phi(y)|\Omega\rangle$$

$$= \frac{\langle 0|T\,\phi_I(x)\,\phi_I(y)\exp\left[i\int d^4z\,\mathcal{L}_I(z)\right]|0\rangle}{\langle 0|T\exp\left[i\int d^4z\,\mathcal{L}_I(z)\right]|0\rangle}$$

$$= \frac{\langle 0|T\,\phi_I(x)\,\phi_I(y)|0\rangle_{\text{connected}}\times\exp\left(\sum_i B_i\right)}{\exp\left(\sum_i B_i\right)}$$

$$= \sum\left(\begin{array}{c}\text{all connected Feynman diagrams with 2}\\ \text{external vertices (no vacuum bubbles)}\end{array}\right)$$

Result generalizes to any n-point function:

$$\langle \Omega | T \phi(x_1)\, \phi(x_2) \ldots \phi(x_n) | \Omega \rangle$$

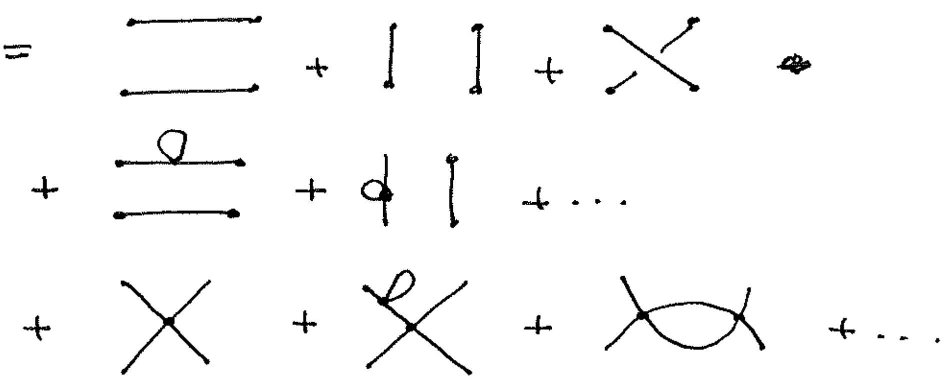
$$= \langle 0 | T \phi_I(x_1) \ldots \phi_I(x_n) | 0 \rangle_{\text{connected}}$$

$$= \sum \begin{pmatrix} \text{all connected diagrams with} \\ n \text{ external points} \end{pmatrix}$$

example: 4-point function

$$\langle \Omega | T \phi(x_1) \phi(x_2) \phi(x_3) \phi(x_4) | \Omega \rangle$$

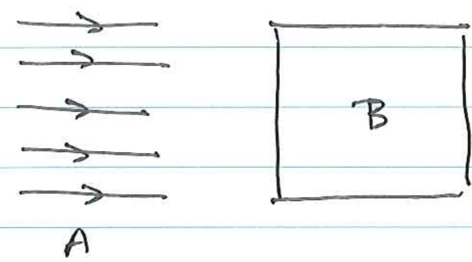
=



note: <u>connected</u> means connected to external points.
i.e. no subdiagram disconnected from $x_1, .., x_n$
although subdiagrams can be disconnected
from each other.

Cross sections, decay rates, and the S-matrix

Consider flux of particles A colliding with a target of particles B.



idealized case: continuous (time-independent) flux, particles A & B described by plane waves, infinite volume.

Scattering: $AB \to$ final state

$$\frac{\text{Scattering rate}}{\text{volume}} = \frac{\text{prob. of scattering}}{\text{volume}\cdot\text{time}} = \sigma \, \Phi_A \underbrace{n_B}_{\text{number density}}$$

$$\Phi_A = \text{flux of } A = n_A v_{rel}.$$

$$\frac{\text{Prob}(AB \to \text{final})}{V \cdot T} = \underbrace{\sigma}_{\text{cross section}} \underbrace{n_A n_B v_{rel}}_{\text{depends on experimental setup}}.$$

Scattering probability is related to the S-matrix.

$$S_{fi} = \underbrace{{}_{out}\langle f}_{\substack{\text{final state}\\ \text{defined at}\\ \text{time } T\to\infty}} | \underbrace{i\rangle_{in}}_{\substack{\text{initial state}\\ \text{defined at}\\ \text{time } T\to -\infty}} = \lim_{T\to\infty} \langle f | \underbrace{e^{-iH(2T)}}_{S} | i \rangle$$

evaluated at a common reference time

The probability $P(i \to f) = |S_{fi}|^2$

$$\sum_f P(i \to f) = 1 = \sum_f S_{fi} S_{fi}^* = (S S^\dagger)_{ii}$$

Unitarity of S-matrix $\Longleftrightarrow$ probabilities add to 1.

Convenient to write:

$$S = \mathbb{1} + iT \qquad \text{or} \qquad S_{fi} = \delta_{fi} + i T_{fi}$$

$\mathbb{1}$ corresponds to no scattering (particles miss each other, or no interaction) $i = f$.
T-matrix describes interactions.

T-matrix will respect 4-momentum conservation $\sum p_i = \sum p_f$, so useful to write:

$$i T_{fi} = i \langle f | T | i \rangle = (2\pi)^4 \delta^4(\sum p_i - \sum p_f)\, i \mathcal{M}(i \to f)$$

$\mathcal{M}$ is called the <u>matrix element</u>.

Now, consider $AB \to 1\,2 \dots n$ scattering, where $A, B, 1, \dots, n$ are all described by plane wave states.

$$|i\rangle = |p_A, p_B\rangle$$
$$|f\rangle = |p_1, p_2, \dots, p_n\rangle$$

Assume $i \neq f$ so $S_{fi} = i T_{fi}$

$$\frac{\text{Prob}(AB \to 12 \cdots n)}{VT} = \frac{1}{VT} \int \text{phase space} \ \frac{|\langle i|T|f\rangle|^2}{\langle i|i\rangle\langle f|f\rangle} \times \underbrace{N_A N_B}_{\substack{\text{number of} \\ A \,\&\, B \text{ particles}}}$$

takes care of states not being normalized to unity.

$$\langle i|i\rangle = \langle p_A|p_A\rangle\langle p_B|p_B\rangle$$
$$= 2E_A\, 2E_B \left((2\pi)^3 \delta^3(0)\right)^2 = 2E_A\, 2E_B\, V^2$$
$$\langle f|f\rangle = (2E_1)(2E_2) \cdots (2E_n)\, V^n$$

$\int$ phase space = integrate over phase space $\int d^3x \int d^3p$ for each final state particle $1, 2, \ldots, n$

$$= V^n \int \frac{d^3p_1}{(2\pi)^3} \int \frac{d^3p_2}{(2\pi)^3} \cdots \int \frac{d^3p_n}{(2\pi)^3}$$

$$|\langle i|T|f\rangle|^2 = |\mathcal{M}(i \to f)|^2 (2\pi)^4 \delta^4(p_A + p_B - p_1 - \cdots - p_n)$$
$$\times \underbrace{(2\pi)^4 \delta^4(0)}_{VT}$$

$$\frac{\text{Prob}(AB \to 12 \cdots n)}{VT} = \frac{1}{VT} \frac{V^n}{V^{n+2}} N_A N_B \int \frac{d^3p_1}{(2\pi)^3} \cdots \int \frac{d^3p_n}{(2\pi)^3} (2\pi)^4 \delta^4(p_A + p_B - \cdots)$$
$$\times |\mathcal{M}(AB \to 12 \cdots n)|^2\, VT$$

$$= \sigma\, v_{rel} \frac{N_A N_B}{V^2}.$$

$\Rightarrow$

$$\boxed{\sigma(AB \to 12 \cdots n) = \frac{1}{4E_A E_B\, v_{rel}} \int \frac{d^3p_1}{(2\pi)^3} \frac{1}{2E_1} \cdots \int \frac{d^3p_n}{(2\pi)^3} \frac{1}{2E_n} (2\pi)^4 \delta^4(p_A + p_B - p_1 - \cdots - p_n) \times |\mathcal{M}(AB \to 12 \cdots n)|^2}$$

exclusive $\sigma(AB \to f)$

inclusive $\sum_f \sigma(AB \to f) = \sigma(AB \to \text{anything})$

Note: σ is Lorentz invariant with respect to boosts along the "beam" direction (direction of $\underline{v}_{rel} = \underline{v}_A - \underline{v}_B$)

Take beam direction along $\hat{z}$ axis:

$$E_A E_B v_{rel} = E_A E_B \left| \frac{p_A^z}{E_A} - \frac{p_B^z}{E_B} \right| = \left| E_B p_A^z - E_A p_B^z \right|$$

$$= \Big| (\gamma E_B' + \beta\gamma p_B^{z\prime})(\gamma p_A^{z\prime} + \gamma\beta E_A')$$

$$- (\gamma E_A' + \beta\gamma p_A^{z\prime})(\gamma p_B^{z\prime} + \gamma\beta E_B') \Big|$$

$$= \left| E_B' p_A^{z\prime} - E_A' p_B^{z\prime} \right| = E_A' E_B' v_{rel}'$$

Useful frames:

"CM frame": $\underline{p}_A = -\underline{p}_B$

"lab frame": $\underline{p}_B = 0$.

<u>Decay rates</u>: Consider an unstable particle A decaying $A \to 12 \cdots n$.

idealized case: constant uniform density of A particles.

$$\frac{\text{Prob}(A \to 12..n)}{\text{Vol.} \cdot \text{Time}} = n_A \cdot P(A \to 12..n)$$

$$= \frac{1}{VT} \int_{\text{phase space}} \frac{|\langle i|T|f\rangle|^2}{\langle i|i\rangle\langle f|f\rangle} \cdot N_A$$

$$= \frac{1}{VT} \int \frac{d^3p_1}{(2\pi)^3} \cdots \int \frac{d^3p_n}{(2\pi)^3} V^n \frac{1}{V^{n+1}} \frac{1}{2E_A} \frac{1}{2E_1} \cdots \frac{1}{2E_n}$$

$$\times |\mathcal{M}(i \to f)|^2 (2\pi)^4 \delta^4(p_A - \Sigma p_f) \underbrace{(2\pi)^4 \delta^4(0)}_{VT} N_A$$

$$\boxed{\begin{aligned}\Gamma(A\to 1\cdots n) = \frac{1}{2m_A}\int\frac{d^3p_1}{(2\pi)^3}\frac{1}{2E_1}\cdots\int\frac{d^3p_n}{(2\pi)^3}\frac{1}{2E_n}(2\pi)^4\delta^4(p_A-p_1-\cdots-p_n)\\ \times\,|\mathcal{M}(A\to 12\ldots n)|^2\end{aligned}}$$

for A at rest.

if A is boosted $\frac{1}{2m_A} \to \frac{1}{2E_A} = \frac{1}{2m_A\gamma}$

$$\Gamma(A\to 1\ldots n) \to \frac{1}{\gamma}\Gamma(A\to 1\ldots n)$$

Decay rate is time-dilated in usual way.

~~Matrix elements from Feynman diagrams~~

$\Gamma(A\to f)$ is called "partial decay rate"

$\Gamma = \sum_f \Gamma(A\to f)$ is "total decay rate"

lifetime of A is $\tau = \frac{1}{\Gamma}$.

Next, relate S-matrix to n-point correlation function
$\Rightarrow$ calculate $\mathcal{M}$ using Feynman diagrams.

Consider 2-point function:
Note: for free fields $\int d^4x\, e^{ip\cdot x} \langle 0 | T\, \phi_I(x)\, \phi_I(0) | 0 \rangle$

$$= \int d^4x\, e^{ip\cdot x} D_F(x) = \frac{i}{p^2 - m^2 + i\varepsilon}$$

$$= \int d^4x\, e^{ip\cdot x}$$ 

Will show for interacting fields

$$\int d^4x\, e^{ip\cdot x} \langle \Omega | T\, \phi(x)\, \phi(0) | \Omega \rangle$$

$$= \frac{iZ}{p^2 - m_*^2 + i\varepsilon} + \ldots$$ terms non-resonant for $p^2 = m_*^2$

$$= \int d^4x\, e^{ip\cdot x}$$ 

$$= \int d^4x\, e^{ip\cdot x}$$ 

$m_*^2 - m^2$ = shift in mass from interactions
(m_*^2 = physical mass, m^2 = parameter in $\mathcal{L}$)
Z = field strength renormalization.

LSZ (Lehmann, Symanzik, Zimmermann) reduction formula:

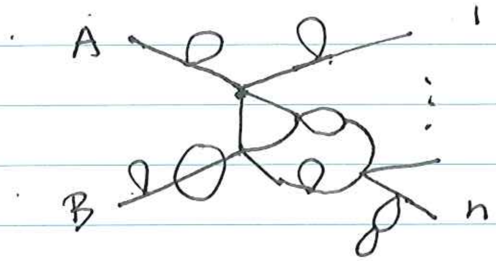
$$\int d^4x_1\, e^{ip_1\cdot x_1} \cdots \int d^4x_n\, e^{ip_n\cdot x_n} \int d^4x_A\, e^{-ip_A\cdot x_A} \int d^4x_B\, e^{-ip_B\cdot x_B}$$
$$\times \langle \Omega | T\, \phi(x_1)\, \phi(x_2) \cdots \phi(x_n)\, \phi(x_A)\, \phi(x_B) | \Omega \rangle$$

$$= \frac{\sqrt{Z}\, i}{p_1^2 - m_*^2 + i\varepsilon} \cdots \frac{\sqrt{Z}\, i}{p_B^2 - m_*^2 + i\varepsilon} \langle p_1 \cdots p_n | S | p_A p_B \rangle$$

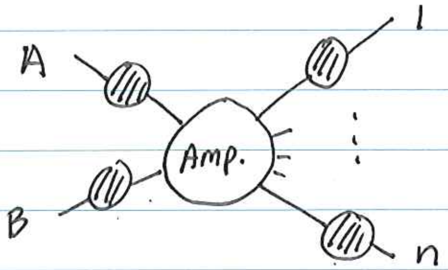
in the limit that $p_1^2 \to m_*^2, \ldots, p_B^2 \to m_*^2$

Proof in PS ch 7.

Consider an arbitrary Feynman diagram contributing to $\langle \Omega | T \phi(x_1) \cdots \phi(x_B) | \Omega \rangle$. $(n+2)$ external points.

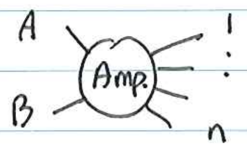


LSZ formula implies that all contributions factorize as:



external leg diagrams:

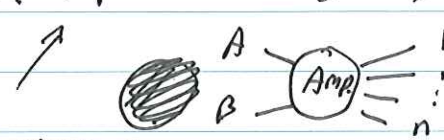
can be "amputated" by cutting one line



$= \Sigma$ (all diagrams with external leg corrections amputated off)

external leg corrections cancel on LHS up to factors of $(\sqrt{Z})^{n+2}$

$$\Rightarrow \langle p_1 \cdots p_n | S | p_A p_B \rangle = (\sqrt{Z})^{n+2} \, \Sigma (\text{amputated diagrams})$$



with external momenta $p_A, p_B, p_1, \ldots$

Note: $Z = 1 + \mathcal{O}(\lambda^2)$ so can neglect at lowest order in perturbation theory example: $2 \to 2$ scattering in ϕ^4-theory ($A, B, 1, 2$ are ϕ particles)

$$\phi(p_A)\, \phi(p_B) \longrightarrow \phi(p_1)\, \phi(p_2)$$

$\mathcal{O}(\lambda^0)$ contribution:

$${}_0\langle p_1 p_2 | p_A p_B \rangle_0 = \sqrt{2E_1}\sqrt{2E_2}\sqrt{2E_A}\sqrt{2E_B}\, \langle 0 | a_{p_1} a_{p_2} a^\dagger_{p_A} a^\dagger_{p_B} | 0 \rangle$$

$$= \sqrt{2E_1}\sqrt{2E_2}\sqrt{2E_A}\sqrt{2E_B}\, \langle 0 | N\big(a_{p_1} a_{p_2} a^\dagger_{p_A} a^\dagger_{p_B} + \text{all possible contractions}\big) | 0 \rangle$$

$$= \sqrt{2E_1}\sqrt{2E_2}\sqrt{2E_A}\sqrt{2E_B}\, \langle 0 | a_{p_1} a_{p_2} a^\dagger_{p_A} a^\dagger_{p_B} + a_{p_1} a_{p_2} a^\dagger_{p_A} a^\dagger_{p_B} | 0 \rangle$$

where e.g. $a_{p_1} a^\dagger_{p_A} = [a_{p_1}, a^\dagger_{p_A}] = (2\pi)^3 \delta^3(p_1 - p_A)$

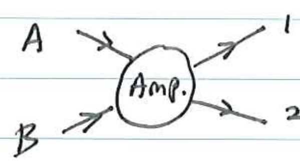
$${}_0\langle p_1 p_2 | p_A p_B \rangle_0 = 4E_A E_B (2\pi)^3 \big[\delta^3(p_1 - p_A)\,\delta^3(p_2 - p_B) + \delta^3(p_1 - p_B)\,\delta^3(p_2 - p_A)\big]$$

also denote this as: $= {}_0\langle p_1 p_2 | p_A p_B \rangle_0 + {}_0\langle p_1 p_2 | p_A p_B \rangle$

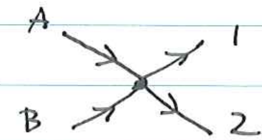
where $\langle \cdots p \cdots | \cdots q \cdots \rangle = \langle \cdots | \cdots \rangle \langle p | q \rangle = \langle \cdots | \cdots \rangle\, 2E_p (2\pi)^3 \times \delta^3(p - q)$

Note: this is the "1" term in the S-matrix
no scattering occurs.

$\mathcal{O}(\lambda)$ contribution:



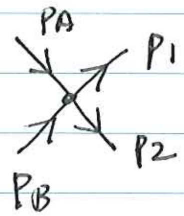
$=$



$= -i\lambda\,(2\pi)^4\,\delta^4(p_A+p_B-p_1-p_2)$

using momentum space feynman rules.
(symmetry factor = 1)

$\Rightarrow \; i\mathcal{M} = -i\lambda.$



$= {}_0\langle p_1 p_2 |\, T\, i\int d^4z\, \mathcal{L}_I(z)\, | p_A p_B\rangle_0$

$= -\dfrac{i\lambda}{4!}\int d^4z\; {}_0\langle p_1 p_2 | T\phi_I(z)^4 | p_A p_B\rangle_0$

$= -\dfrac{i\lambda}{4!}\int d^4z\, \sqrt{2E_1}\sqrt{2E_2}\sqrt{2E_A}\sqrt{2E_B}\; \underbrace{\langle 0 | a_{p_1} a_{p_2}\, \phi_I(z)^4\, a^\dagger_{p_A} a^\dagger_{p_B} | 0\rangle}_{\text{Σ permutations with all terms contracted}}$

e.g.

$\langle 0 | a_{p_1} a_{p_2}\, \phi\, \phi\, \phi\, \phi\, a^\dagger_{p_A}\, a^\dagger_{p_B} | 0\rangle$

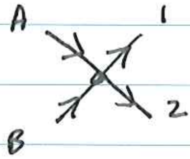
$= -\dfrac{i\lambda}{4!}\int d^4z\; {}_0\langle p_1 p_2 |\, \phi_I(z)\, \phi_I(z)\, \phi_I(z)\, \phi_I(z)\, | p_A p_B\rangle$

$\times$ (4! possible contractions)

$\phi_I(z)\,|p\rangle = \int \dfrac{d^3q}{(2\pi)^3}\dfrac{1}{\sqrt{2E_q}}\, a_q\, e^{-iq\cdot z}\, a^\dagger_p \sqrt{2E_p}\,|0\rangle$

$= e^{-ip\cdot z}\,|0\rangle$

$$\langle p | \phi_I(z) = \int \frac{d^3q}{(2\pi)^3} \frac{1}{\sqrt{2E_q}} \langle 0 | a_p \sqrt{2E_p}\, a_q^\dagger \; e^{iq\cdot z} = \langle 0 | e^{iq\cdot z}$$



$$= \frac{-i\lambda}{4!} \times 4! \int d^4z\, e^{-i(p_A+p_B-p_1-p_2)\cdot z}$$

$$= \underbrace{-i\lambda}_{= i\mathcal{M}} (2\pi)^4 \delta^4(p_A+p_B-p_1-p_2)$$

Matrix element: $\mathcal{M} = -\lambda$.

Compute cross section:

$$\sigma = \frac{1}{4E_AE_B v_{rel}} \int \frac{d^3p_1}{(2\pi)^3} \frac{1}{2E_1} \int \frac{d^3p_2}{(2\pi)^3} \frac{1}{2E_2} (2\pi)^4 \delta^4(p_A+p_B-p_1-p_2) \cdot \frac{1}{2} |\mathcal{M}|^2$$

due to identical particles in final state.

2-body phase space integral:

$$I_2 = \int \frac{d^3p_1}{(2\pi)^3} \frac{1}{2E_1} \int \frac{d^3p_2}{(2\pi)^3} \frac{1}{2E_2} (2\pi)^4 \delta^4(p_A+p_B-p_1-p_2)$$

work in CM frame: $\vec{p}_A = -\vec{p}_B \rightarrow \vec{p}_1 = -\vec{p}_2$.

$E_A = E_B \equiv \frac{E_{cm}}{2} \rightarrow E_1 = E_2 = \frac{E_{cm}}{2}$

for identical particles $A=B$ and $1=2$.

First, do $\int d^3p_2$ integral using $\delta^3(\ldots)$:

$$I_2 = \int \frac{d^3p_1}{(2\pi)^3} \frac{1}{4E_1E_2} (2\pi)\, \delta(E_A+E_B-E_1-E_2)$$

where $E_2 = \sqrt{|\vec{p}_1|^2+m^2} = E_1$

$$\delta(E_A + E_B - E_1 - E_2) = \delta(E_{cm} - 2E_1) = \tfrac{1}{2}\delta\!\left(E_1 - \tfrac{E_{cm}}{2}\right)$$

$$I_2 = \int \frac{d^3p_1}{(2\pi)^3} \frac{1}{8E_1E_2} (2\pi)\, \delta\!\left(E_1 - \frac{E_{cm}}{2}\right)$$

next, do $\int d^3p_1$ integral:

$$\int d^3p_1 = \int_0^\infty p_1^2\, dp_1 \underbrace{\int_{-1}^{+1} d\cos\theta \int_0^{2\pi} d\phi}_{\int d\Omega} = \int p_1^2\, dp_1 \int d\Omega$$

note: $E_1^2 = p_1^2 + m^2$

$$2E_1 dE_1 = 2p_1 dp_1 \rightarrow p_1 dp_1 = E_1 dE_1$$

$$\int d^3p_1\, \delta\!\left(E_1 - \frac{E_{cm}}{2}\right) = \int p_1 E_1\, dE_1 \int d\Omega\, \delta\!\left(E_1 - \frac{E_{cm}}{2}\right)$$

$$= p_1 \frac{E_{cm}}{2} \int d\Omega \qquad \text{where } p_1 = \sqrt{\frac{E_{cm}^2}{4} - m^2}$$

$$I_2 = \frac{1}{(2\pi)^3} \frac{1}{2E_{cm}^2} (2\pi)\, p_1 \frac{E_{cm}}{2} \int d\Omega = \frac{1}{16\pi^2} \frac{p_1}{E_{cm}} \int d\Omega$$

$$= \frac{p_1}{16\pi^2 E_{cm}} \int d\Omega$$

can't do $\int d\Omega$ in general since $\mathcal{M}$ can depend on scattering angle.

$$\sigma = \frac{1}{E_{cm}^2 v_{rel}} \left(\frac{p_1}{16\pi^2 E_{cm}} \int d\Omega \right) \frac{1}{2} |\mathcal{M}|^2$$

$$v_{rel} = \left| \frac{p_A}{E_A} - \frac{p_B}{E_B} \right| = \frac{2p_A}{E_A} = 2\sqrt{1 - m^2/E_A^2} = 2\sqrt{1 - \frac{4m^2}{E_{cm}^2}}$$

also: $\dfrac{p_1}{E_{cm}} = \dfrac{1}{2}\sqrt{1 - \dfrac{4m^2}{E_{cm}^2}} = \dfrac{v_{rel}}{4}$

So $$\sigma = \int d\Omega \frac{\lambda^2}{128\pi^2 E_{cm}^2} = \frac{\lambda^2}{32\pi E_{cm}^2}$$

differential cross section:

$$\frac{d\sigma}{d\Omega} = \frac{\lambda^2}{64\pi^2 E_{cm}^2}$$

don't need to multiply by $\frac{1}{2}$ for $\frac{d\sigma}{d\Omega}$ since $\phi(p_1)\ \phi(p_2)$ have distinct momenta.