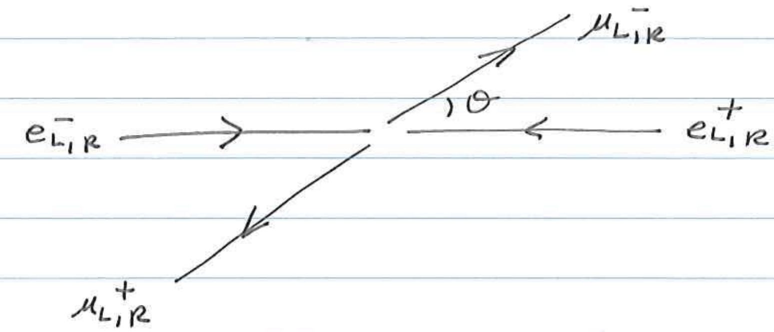
## Helicity structure of $e^+e^- \to \mu^+\mu^-$

Next, consider polarized $e^+e^- \to \mu^+\mu^-$ scattering.
Work in relativistic limit: $E_{cm} \gg m_\mu, m_e$.

Each fermion has two spin (helicity) states: L, R
→ 16 possible combinations




matrix element:

$$i\mathcal{M} = \frac{ie^2}{q^2}\,\bar{v}(p')\gamma^\mu u(p)\;\;\bar{u}(k)\gamma_\mu v(k')$$

Trick: use projection operators $P_L = \frac{1-\gamma^5}{2}$ and $P_R = \frac{1+\gamma^5}{2}$ to select desired helicity states.

incoming $e^-_{L,R}$: $\quad u(p) \to P_{L,R}\,u(p)$
outgoing $\mu^-_{L,R}$: $\quad \bar{u}(k) \to \bar{u}(k)\,P_{R,L} \qquad$ (since $u^\dagger P_{L,R}\gamma^0 = \bar{u}P_{R,L}$)

outgoing $\mu^+_{L,R}$: $\quad v(k') \to P_{R,L}\,v(k')$
incoming $e^+_{L,R}$: $\quad \bar{v}(p') \to \bar{v}(p')\,P_{L,R}$

$e^-_{L,R}$ : $u(p) \rightarrow P_{L,R}\, u(p)$

$\mu^-_{L,R}$ : $\bar{u}(k) \rightarrow \bar{u}(k)\, P_{R,L}$ (since $\gamma^0 P_{L,R} = P_{R,L}\gamma^0$)

$\mu^+_{L,R}$ : $v(k') \rightarrow P_{R,L}\, v(k')$

$e^+_{L,R}$ : $\bar{v}(p') \rightarrow \bar{v}(p')\, P_{L,R}$

So

$$i\mathcal{M}(e^-_L e^+_R \rightarrow \mu^-_L \mu^+_R) = \frac{ie^2}{q^2}\bar{v}(p') P_R \gamma^\mu P_L u(p)\ \ \bar{u}(k) P_R \gamma_\mu P_L v(k')$$

$$= \frac{ie^2}{q^2}\bar{v}(p')\gamma^\mu P_L u(p)\ \ \bar{u}(k)\gamma_\mu P_L v(k')$$

Can compute $|\mathcal{M}|^2$ by summing over spins since $P_{L,R}$ already project out desired spins.

$$|\mathcal{M}(e^-_L e^+_R \rightarrow \mu^-_L \mu^+_R)|^2 = \sum_{s,s',r,r'} |\mathcal{M}(e^-_L e^+_R \rightarrow \mu^-_L \mu^+_R)|^2$$

$$= \frac{e^4}{q^4} \mathrm{Tr}[\not{p}'\gamma^\mu P_L \not{p}\gamma^\nu P_L]\ \mathrm{Tr}[\not{k}\gamma_\mu P_L \not{k}'\gamma_\nu P_L]$$

$$\mathrm{Tr}[\not{p}'\gamma^\mu P_L \not{p}\gamma^\nu P_L] = \frac{1}{2}\mathrm{Tr}[\not{p}'\gamma^\mu \not{p}\gamma^\nu(1-\gamma^5)]$$

$$= 2(p'^\mu p^\nu + p^\mu p'^\nu - g^{\mu\nu} p\cdot p' + i\varepsilon^{\alpha\mu\beta\nu} p'_\alpha p_\beta)$$

$$\mathrm{Tr}[\not{k}\gamma_\mu P_L \not{k}'\gamma_\nu P_L] = 2(k_\mu k'_\nu + k_\nu k'_\mu - k\cdot k' g_{\mu\nu} + i\varepsilon^{\sigma\mu\tau\nu} k_\sigma k'_\tau)$$

$$|\mathcal{M}(e_L^- e_R^+ \to \mu_L^- \mu_R^+)|^2 = \frac{4e^4}{q^4}\Big(2p\cdot k\, p'\cdot k' + 2p\cdot k'\, p'\cdot k$$
$$- \varepsilon^{\alpha\mu\beta\nu}\varepsilon_{\sigma\mu\tau\nu}\, p'_\alpha p_\beta k_\sigma k'_\tau\Big)$$

Note: $\varepsilon^{\mu\nu\alpha\beta}\,\varepsilon_{\mu\nu\sigma\tau} = -2\left(g^\alpha{}_\sigma\, g^\beta{}_\tau - g^\alpha{}_\tau\, g^\beta{}_\sigma\right)$

$$\varepsilon^{\mu\nu\alpha\beta}\,\varepsilon_{\mu\nu\alpha\tau} = -6\, g^\beta{}_\tau$$

$$\varepsilon^{\mu\nu\alpha\beta}\,\varepsilon_{\mu\nu\alpha\beta} = -24 = -4!$$

$$|\mathcal{M}(e_L^- e_R^+ \to \mu_L^- \mu_R^+)|^2 = \frac{4e^4}{q^4}\Big(2p\cdot k\, p'\cdot k' + 2p\cdot k'\, p'\cdot k$$
$$+ 2p'\cdot k\; p\cdot k' - 2p'\cdot k'\; p\cdot k\Big)$$

$$= \frac{16 e^4}{q^4}\left(p\cdot k'\; p'\cdot k\right)$$

$$= \frac{16 e^4}{16E^4}\, E^4 (1+\cos\theta)^2$$

$$= e^4 (1+\cos\theta)^2$$

$$|\mathcal{M}(e_R^- e_L^+ \to \mu_R^- \mu_L^+)|^2 = e^4 (1+\cos\theta)^2$$

$$|\mathcal{M}(e_L^- e_R^+ \to \mu_R^- \mu_L^+)|^2 = e^4 (1-\cos\theta)^2$$

$$|\mathcal{M}(e_R^- e_L^+ \to \mu_L^- \mu_R^+)|^2 = e^4 (1-\cos\theta)^2$$

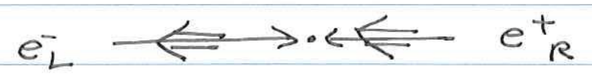
$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2 E_{cm}^2}|\mathcal{M}|^2$$

$e_L^- e_R^+ \to \mu_L^- \mu_R^+$, $e_R^- e_L^+ \to \mu_R^- \mu_L^+$ $\qquad \dfrac{d\sigma}{d\Omega} = \dfrac{\alpha^2}{4E_{cm}^2}(1+\cos\theta)^2$

$e_L^- e_R^+ \to \mu_R^- \mu_L^+$, $e_R^- e_L^+ \to \mu_L^- \mu_R^+$ $\qquad \dfrac{d\sigma}{d\Omega} = \dfrac{\alpha^2}{4E_{cm}^2}(1-\cos\theta)^2$

12 other possibilities: $\dfrac{d\sigma}{d\Omega} = 0$

Angular momentum conservation: ($e_L^- e_R^+ \to \mu_L^- \mu_R^+$)




$J_z = -1$


$J_z = -1$ $\qquad \dfrac{d\sigma}{d\Omega} \neq 0$

$\theta = 0$ ($\cos\theta = 1$)

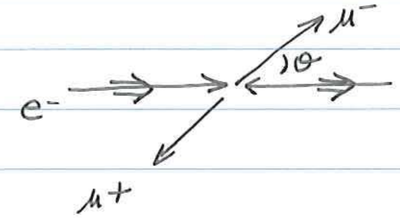

$J_z = +1 \quad \to \quad \dfrac{d\sigma}{d\Omega} = 0$

$\theta = \pi$ ($\cos = -1$)

$(1 \pm \cos\theta)$ - dependence ensures initial & final states have same $J_z$.

Also, note only opposite helicity initial and final states are allowed (e.g. $e_{L,R}^- e_{R,L}^+$ but not $e_{L,R}^- e_{L,R}^+$). Intermediate photon state must have polarization $J_z = \pm 1$.

Also useful to evaluate matrix elements directly:





e.g. $e^-_R e^+_L \to \mu^-_R \mu^+_L$

$$u(p) = \begin{pmatrix} \sqrt{p\cdot\sigma}\,\xi_s \\ \sqrt{p\cdot\bar\sigma}\,\xi_s \end{pmatrix} = \sqrt{2E}\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \sqrt{E - p^3\sigma^3}\,\xi_s \\ \sqrt{E + p^3\sigma^3}\,\xi_s \end{pmatrix}$$

$\xi_s = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ spin up along $\hat{z}$ axis

$$v(p') = \begin{pmatrix} \sqrt{p'\cdot\sigma}\,\eta_s \\ -\sqrt{p'\cdot\bar\sigma}\,\eta_s \end{pmatrix} = \begin{pmatrix} \sqrt{E + p^3\sigma^3}\,\eta_s \\ -\sqrt{E - p^3\sigma^3}\,\eta_s \end{pmatrix} = \sqrt{2E}\begin{pmatrix} 0 \\ 0 \\ 0 \\ -1 \end{pmatrix}$$

using $p'^3 = -p^3$

$\eta_s = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$
spin up along $\hat{z}$ axis.

So
$$\bar{v}(p')\gamma^\mu u(p) = 2E\,(0,0,0,-1)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma^\mu \\ \bar\sigma^\mu & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

$$= 2E\,(0,0,0,-1)\begin{pmatrix} \bar\sigma^\mu & 0 \\ 0 & \sigma^\mu \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \cancel{= 2E\sigma^\mu}$$

$$= +2E\,(0,-1)\,\sigma^\mu \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \cancel{-}2E\,(0,1,i,0)^\mu$$

$$= \underbrace{-2E\,(0,1,i,0)^\mu}_{\text{circular polarization vector } \varepsilon = \hat{x} + i\hat{y}}$$

Interpretted as polarization of intermediate photon with angular momentum $J_z = +$.

muon term: $\bar{u}(k)\gamma_\mu v(k') = \left(\bar{v}(k')\gamma_\mu u(k)\right)^*$

Just rotate previous result by angle $\theta$ in $\hat{x}$-$\hat{z}$ plane.
and complex conjugate.

$$\bar{u}(k)\gamma_\mu v(k') = -2E\,(0, \cos\theta, -i, -\sin\theta)_\mu$$

$$\text{So } \mathcal{M}(e_R^+ e^- \to \mu_R^- \mu_L^+) = \frac{e^2}{q^2}\,\bar{v}(p')\gamma^\mu u(p)\;\bar{u}(k)\gamma_\mu v(k')$$

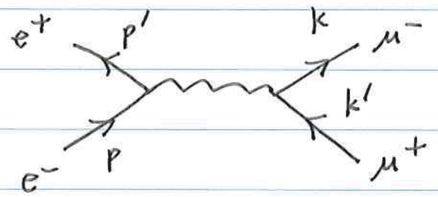
$$= -e^2(1+\cos\theta)$$

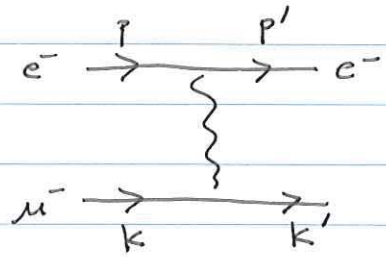
polarization of intermediate photon of $\underline{\varepsilon}' = \hat{n} + i\hat{y}$
where $\hat{n} = \hat{x}\cos\theta - \sin\theta\,\hat{z}$

$\mathcal{M} \propto \underline{\varepsilon}\cdot\underline{\varepsilon}'$ overlap of polarization vectors corresponding to initial & final fermion configurations

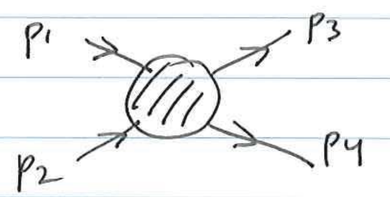
<u>Electron-muon scattering</u>

$e^-\mu^- \to e^-\mu^-$ : related to $e^-e^+ \to \mu^+\mu^-$ by <u>crossing symmetry</u> (exchanging initial & final state particles)




General $2 \to 2$ scattering:



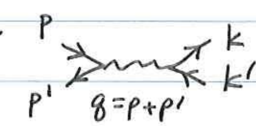
Initial & final states can be swapped by letting $p_i \to -p_i$ in the matrix element.

Mandelstam variables:

$$s = (p_1 + p_2)^2 = (p_3 + p_4)^2$$
$$t = (p_1 - p_3)^2 = (p_2 - p_4)^2$$
$$u = (p_1 - p_4)^2 = (p_2 - p_3)^2$$

easy to check: $s + t + u = m_1^2 + m_2^2 + m_3^2 + m_4^2$

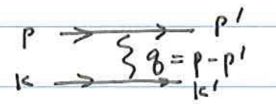
$e^- e^+ \to \mu^- \mu^+$: (let $m_e, m_\mu = 0$)




$$\sum_{\text{spins}} |\mathcal{M}|^2 = \frac{8e^4}{q^4}\left((2p\cdot k)(2p'\cdot k') + (2p\cdot k')(2p'\cdot k)\right)$$

$$= \frac{8e^4}{s^2}\left(t^2 + u^2\right)$$

$e^- \mu^- \to e^- \mu^-$:



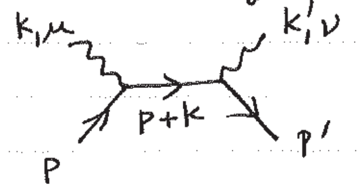

$\swarrow$ $p' \to -p'$, $k' \to -k$, $k \to k'$

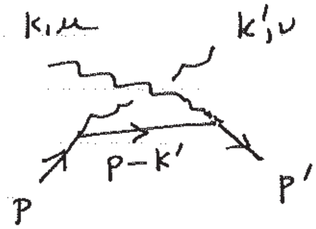
$$\sum_{\text{spins}} |\mathcal{M}|^2 = \frac{8e^4}{q^4}\left((2p\cdot k)(2p'\cdot k') + (2p\cdot k')(2p'\cdot k)\right)$$

$$= \frac{8e^4}{t^2}\left(s^2 + u^2\right)$$

crossing symmetry exchanges $s \leftrightarrow t$.

## Compton scattering

$i\mathcal{M} =$ 

$+$ 


$$= \bar{u}(p')\left[(ie\gamma^\nu)\frac{i(\not{p}+\not{k}+m)}{(p+k)^2-m^2}(ie\gamma^\mu) + (ie\gamma^\mu)\frac{i(\not{p}-\not{k}'+m)}{(p-k')^2-m^2}(ie\gamma^\nu)\right]$$

$$\times\ \bar{u}(p)\ \varepsilon_\mu(k)\ \varepsilon_\nu(k')^*$$

$$= -ie^2\varepsilon_\mu(k)\,\varepsilon_\nu(k')\,\bar{u}(p')\left[\frac{\gamma^\nu(\not{p}+\not{k}+m)\gamma^\mu}{2p\cdot k} + \frac{\gamma^\mu(\not{p}-\not{k}'+m)\gamma^\nu}{-2p\cdot k'}\right]u(p)$$

use Dirac equation $\not{p}\,u(p) = m\,u(p)$

$$\text{Then}\quad (\not{p}+m)\gamma^\mu u(p) = \left[2p^\mu - \gamma^\mu(\not{p}-m)\right]u(p)$$
$$= 2p^\mu u(p)$$

$$i\mathcal{M} = -ie^2\varepsilon_\mu(k)\,\varepsilon_\nu(k')^*\,\bar{u}(p')\left[\frac{\gamma^\nu\not{k}\gamma^\mu + 2\gamma^\nu p^\mu}{2p\cdot k} + \frac{\gamma^\mu\not{k}'\gamma^\nu - 2\gamma^\mu p^\nu}{2p\cdot k'}\right]u(p)$$

$$\sum|\mathcal{M}|^2 = e^4\left(\sum_{pol}\varepsilon_\mu(k)\,\varepsilon_\lambda(k)^*\right)\left(\sum_{pol}\varepsilon_\nu(k')^*\,\varepsilon_\kappa(k')\right)$$
$$\times\ \mathrm{Tr}\left[(\not{p}'+m)\left(\frac{\gamma^\nu\not{k}\gamma^\mu + 2\gamma^\nu p^\mu}{2p\cdot k} + \frac{\gamma^\mu\not{k}'\gamma^\nu - 2\gamma^\mu p^\nu}{2p\cdot k'}\right)\right.$$
$$\left.\times\ (\not{p}+m)\left(\frac{\gamma^\lambda\not{k}\gamma^\kappa + 2\gamma^\kappa p^\lambda}{2p\cdot k} + \frac{\gamma^\kappa\not{k}'\gamma^\lambda - 2\gamma^\lambda p^\kappa}{2p\cdot k'}\right)\right]$$

use rule: $\sum \varepsilon_\mu\,\varepsilon_\lambda^* \to -g_{\mu\lambda}$

$$\sum |\mathcal{M}|^2 = e^4 \,\mathrm{Tr}\left[(\not{p}'+m)\left(\frac{\gamma^\nu \not{k}\gamma^\mu + 2\gamma^\nu p^\mu}{2p\cdot k} + \frac{\gamma^\mu \not{k}'\gamma^\nu - 2\gamma^\mu p^\nu}{2p\cdot k'}\right)\right.$$
$$\left. \times (\not{p}+m)\left(\frac{\gamma_\mu \not{k}\gamma_\nu + 2\gamma_\nu p_\mu}{2p\cdot k} + \frac{\gamma_\nu \not{k}'\gamma_\mu - 2\gamma_\mu p_\nu}{2p\cdot k'}\right)\right]$$

Dirac matrix identities:

$$\gamma^\mu \gamma_\mu = 4 \qquad \not{p}\not{p} = \tfrac{1}{2}\{\not{p},\not{p}\} = p^2$$

$$\gamma^\mu \gamma^\nu \gamma_\mu = -\gamma^\mu \gamma_\mu \gamma^\nu + 2\gamma^\mu g^\nu{}_\mu = -4\gamma^\nu + 2\gamma^\nu$$
$$= -2\gamma^\nu$$

$$\gamma^\mu \not{p} \gamma_\mu = -2\not{p}$$

$$\gamma^\mu \gamma^\alpha \gamma^\beta \gamma_\mu = -\gamma^\mu \gamma^\alpha \gamma_\mu \gamma^\beta + 2\gamma^\mu \gamma^\alpha g^\beta{}_\mu$$
$$= 2\gamma^\alpha\gamma^\beta + 2\gamma^\beta\gamma^\alpha = 2\{\gamma^\alpha, \gamma^\beta\}$$
$$= 4 g^{\alpha\beta}$$

$$\gamma^\mu \gamma^\alpha \gamma^\beta \gamma^\nu \gamma_\mu = -2\gamma^\nu \gamma^\beta \gamma^\alpha \quad \text{(indices reversed)}$$

$$\text{Trace term } 1 = \mathrm{Tr}\left[(\not{p}'+m)(\gamma^\nu \not{k}\gamma^\mu + 2\gamma^\nu p^\mu)(\not{p}+m) \times (\gamma_\mu \not{k}\gamma_\nu + 2\gamma_\nu p_\mu)\right]$$

$$= \mathrm{Tr}\left[\not{p}'(\gamma^\nu \not{k}\gamma^\mu + 2\gamma^\nu p^\mu)\not{p}(\gamma_\mu \not{k}\gamma_\nu + 2\gamma_\nu p_\mu)\right.$$
$$\left. + m^2(\gamma^\nu \not{k}\gamma^\mu + 2\gamma^\nu p^\mu)(\gamma_\mu \not{k}\gamma_\nu + 2\gamma_\nu p_\mu)\right]$$

$$= \mathrm{Tr}\big[\not p'\gamma^\nu \not k \gamma^\mu \not p \gamma_\mu \not k \gamma_\nu + 2\not p'\gamma^\nu \not k \not p \not p \gamma_\nu$$
$$+ 2\not p'\gamma^\nu \not p \not p \not k \gamma_\nu + 4\not p'\gamma^\nu \not p \gamma_\nu p^2$$
$$+ m^2 ( \gamma^\nu \not k \gamma^\mu \gamma_\mu \not k \gamma_\nu + 2\gamma^\nu \not k \not p \gamma_\nu$$
$$+ 2\gamma^\nu \not p \not k \gamma_\nu + 4\gamma^\nu \gamma_\nu p^2)\big]$$

$$= \mathrm{Tr}\big[4\not p' \not k \not p \not k - 4\not p' \not k m^2 - 4\not p' \not k m^2 - 8m^2 p\cdot p'$$
$$+ m^2 (16k^2 + 8k\cdot p + 8k\cdot p + 16m^2)\big]$$

$$= 16\,(2\,p\cdot k\, p'\cdot k - p\cdot p'\, k^2) - 8\cdot 4\, m^2 p'\cdot k - 32\, m^2 p\cdot p'$$
$$+ 64 m^2 k^2 + 64 m^2 k\cdot p + 64 m^2$$

$$= 32\, p\cdot k\, p'\cdot k - 32 m^2 p'\cdot k - 32 m^2 p\cdot p' + 64 m^2 k\cdot p + 64 m^2$$

$$p\cdot p' = -\tfrac{1}{2}(p-p')^2 + m^2 = -\tfrac{1}{2}(k-k')^2 + m^2$$
$$= \tfrac{1}{2}(2\,k\cdot k' + 2m^2)$$
$$= \tfrac{1}{2}(2k\cdot(p+k-p') + 2m^2) = p\cdot k - p'\cdot k + m^2$$

$$= 32\, p\cdot k\, p'\cdot k + 32 m^2\, p\cdot k + 32 m^4$$

trace term 2 = trace term 3

$$= -16 m^2 p\cdot k + 16 m^2 p'\cdot k - 32 m^4$$

trace term 4 $= 32\, p\cdot k\, p'\cdot k - 32\, p'\cdot k\, m^2 + 32 m^4$

$$s = (p+k)^2 = (p'+k')^2 \Rightarrow p\cdot k = p'\cdot k'$$
$$t = (p-p')^2 = (k-k')^2 \Rightarrow p\cdot p' = k\cdot k'$$
$$u = (p-k')^2 = (p'-k)^2 \Rightarrow p\cdot k' = p'\cdot k$$

Lab frame: $e^-$ initially at rest, $p = (m,0,0,0)$

initial $\gamma$: $k = (\omega, 0, 0, \omega)$

final $\gamma$: $k' = (\omega', \omega'\sin\theta, 0, \omega'\cos\theta)$

$$\text{trace } 1 = 32\, p\cdot k\; p\cdot k' + 32\, p\cdot k\, m^2 + 32 m^4$$
$$= 32 m^2 \omega\omega' + 32 m^3 \omega + 32 m^4$$
$$= 32 m^2 (\omega\omega' + m\omega + m^2)$$

$$\text{trace } 2 = -8 m^3 \omega + 8 m^3 \omega' - 32 m^4$$
$$= -8 m^2 (\omega m - m\omega' + 4m^2) = \text{trace } 3$$

$$\text{trace } 4 = 32 m^2 (\omega\omega' - m\omega' + m^2)$$

denominators $\frac{1}{2p\cdot k} = \frac{1}{2m\omega}$, $\frac{1}{2p\cdot k'} = \frac{1}{2m\omega'}$

$$\frac{1}{4}\sum |\mathcal{M}|^2 = e^4 \left\{ \frac{32 m^2(\omega\omega' + m\omega + m^2)}{16 m^2 \omega^2} - \frac{8 m^2 (\omega m - m\omega' + 4m^2)}{16 m^2 \omega\omega'} \times 2 \right.$$
$$\left. + \frac{32 m^2 (\omega\omega' - m\omega' + m^2)}{16 m^2 \omega'^2} \right\}$$

$$= e^4 \left\{ 2\left(\frac{\omega'}{\omega} + \frac{m}{\omega} + \frac{m^2}{\omega^2}\right) - 2\left(\frac{m}{\omega'} - \frac{m}{\omega} + 4\frac{m^2}{\omega\omega'}\right) \right.$$
$$\left. + 2\left(\frac{\omega}{\omega'} - \frac{m}{\omega'} + \frac{m^2}{\omega'^2}\right) \right\}$$
$$= 2e^4 \left[ \frac{\omega}{\omega'} + \frac{\omega'}{\omega} + 2m\left(\frac{1}{\omega} - \frac{1}{\omega'}\right) + m^2\left(\frac{1}{\omega} - \frac{1}{\omega'}\right)^2 \right]$$

Note: $\omega'$ depends on scattering angle $\theta$.

Compton formula : $\Delta\lambda = \frac{h}{mc}(1-\cos\theta) = \frac{2\pi}{m}(1-\cos\theta)$

where $\lambda = \frac{2\pi}{\omega}$

$$\Rightarrow \quad \frac{1}{\omega'} - \frac{1}{\omega} = \frac{1}{m}(1-\cos\theta)$$

Energy-momentum conservation: $\omega + m = E_{p'} + \omega'$

$\underline{k} = \underline{p}' + \underline{k}'$

$$(\omega + m - \omega')^2 = E_{p'}^2 = |\underline{p}'|^2 + m^2 = |\underline{k} - \underline{k}'|^2 + m^2$$

$$= |\underline{k}|^2 + |\underline{k}'|^2 - 2|\underline{k}||\underline{k}'|\cos\theta + m^2$$

$$\omega^2 + \omega'^2 + m^2 + 2\omega m - 2m\omega' - 2\omega\omega' = \omega^2 + \omega'^2 - 2\omega\omega'\cos\theta + m^2$$

$$\Rightarrow \quad \frac{1}{\omega'} - \frac{1}{\omega} = \frac{1}{m}(1-\cos\theta)$$

$$\frac{1}{4}\sum|\mathcal{M}|^2 = 2e^4\left[\frac{\omega}{\omega'} + \frac{\omega'}{\omega} - 2(1-\cos\theta) + (1-\cos\theta)^2\right]$$

$$= 2e^4\left[\frac{\omega}{\omega'} + \frac{\omega'}{\omega} - 2 + 2\cos\theta + 1 - 2\cos\theta + \cos^2\theta\right]$$

$$= 2e^4\left[\frac{\omega}{\omega'} + \frac{\omega'}{\omega} - \sin^2\theta\right]$$

Phase space integral:

$$I_2 = \int\frac{d^3p'}{(2\pi)^3}\frac{1}{2E_{p'}}\int\frac{d^3k'}{(2\pi)^3}\frac{1}{2E_{k'}}(2\pi)^4\delta^4(p+k-p'-k')$$

$$= \int\frac{d^3k}{(2\pi)^3}\frac{1}{4E_{p'}E_{k'}}(2\pi)\,\delta(E_{cm} - E_{p'} - E_{k'})$$

$$I_2 = \frac{1}{16\pi^2} \int_0^\infty k' \, dk' \, \frac{1}{E_{p'}} \int d\Omega \, \delta(E_{cm} - E_{p'} - k')$$

$$E_{p'} = \sqrt{|\underline{p}'|^2 + m^2} = \sqrt{|\underline{k} - \underline{k}'|^2 + m^2} = \sqrt{|\underline{k}|^2 + |\underline{k}'|^2 - 2|\underline{k}||\underline{k}'|\cos\theta + m^2}$$

$$\frac{\partial}{\partial k'} (E_{cm} - E_{p'} - k') = -1 - \frac{\partial E_{p'}}{\partial k'} = -1 - \frac{1}{2}\frac{1}{E_{p'}}(2k' - 2k\cos\theta)$$

$$= -\frac{2E_p + 2k' - 2k\cos\theta}{2E_{p'}}$$

$$= -\frac{2(\omega + m) - 2\omega\cos\theta}{2E_{p'}}$$

$$= -\frac{\omega(1-\cos\theta) + m}{E_{p'}}$$

$$I_2 = \frac{1}{16\pi^2} \int d\Omega \, \frac{\omega'}{\omega(1-\cos\theta) + m} = \frac{1}{8\pi} \int d\cos\theta \, \underbrace{\frac{\omega'}{\omega(1-\cos\theta) + m}}_{= \omega'^2/m\omega}$$

$$\frac{d\sigma}{d\cos\theta} = \frac{1}{4m\omega} \frac{1}{8\pi} \frac{\omega'^2}{m\omega}$$

$$\times \, 2e^4 \left[ \frac{\omega}{\omega'} + \frac{\omega'}{\omega} - \sin^2\theta \right]$$

$$= \frac{\pi\alpha^2}{m^2} \left(\frac{\omega'^2}{\omega^2}\right) \left(\frac{\omega}{\omega'} + \frac{\omega'}{\omega} - \sin^2\theta\right)$$ Klein-Nishina cross section.

Low-energy limit: $\omega \ll m$.

Then $\omega' = \left(\frac{1}{\omega} + \frac{1}{m}(1 - \cos\theta)\right)^{-1} \approx \left(\frac{1}{\omega}\right)^{-1} = \omega$

$$\frac{d\sigma}{d\cos\theta} = \frac{\pi\alpha^2}{m^2}(1 + \cos^2\theta)$$

$$\sigma = \frac{8\pi\alpha^2}{3m^2}$$ Thompson cross section.

## Ward identity

Consider matrix element for process with incoming $\gamma(\underline{k}, \varepsilon)$:

$i\mathcal{M} = i\mathcal{M}_\mu \varepsilon^\mu(\underline{k})$ can factor out polarization vector.

Ward identity: $\mathcal{M}_\mu k^\mu = 0$. (i.e. replacing $\varepsilon^\mu \to k^\mu$)

proof: $i\mathcal{M} \sim \langle \dots | \dots \int d^4z\, A_\mu(z)\, j^\mu(z) \dots | \gamma(\underline{k}, \varepsilon) \dots \rangle$

$$= \langle \dots | \dots \int d^4z\, \varepsilon_\mu(\underline{k})\, e^{-ik\cdot z}\, j^\mu(z) \dots | \dots \rangle$$

$$= \varepsilon_\mu \underbrace{\langle \dots | \dots \int d^4z\, e^{-ik\cdot z}\, j^\mu(z) \dots | \dots \rangle}_{\sim\, i\mathcal{M}_\mu}$$

Now let $\varepsilon_\mu \to k_\mu$:

$$k^\mu \mathcal{M}_\mu \sim \langle \dots | \dots \int d^4z\, k_\mu\, e^{-ik\cdot z}\, j^\mu(z) \dots | \dots \rangle$$

$k \to i\partial_z$ then int. by parts

$$\sim \langle \dots | \dots \int d^4z\, e^{-ik\cdot z}(-\partial_\mu j^\mu) \dots | \dots \rangle = 0$$

since $\partial_\mu j^\mu = 0$ by current conservation.

Now we can prove polarization sum rule: $\sum_{pol} \varepsilon^\mu(\underline{k})\, \varepsilon^\nu(\underline{k}) \to -\eta^{\mu\nu}$

Consider $\underline{k} = k\hat{z}$. Physical polarizations have $\varepsilon \cdot k = 0$.

e.g. transverse polarizations $\varepsilon_1 = (0,1,0,0)$, $\varepsilon_2 = (0,0,1,0)$

unphysical polarizations can be taken as

$\varepsilon_0 = (1,0,0,0)$ and $\varepsilon_3 = (0,0,0,1)$

$$k^\mu \mathcal{M}_\mu = k^0 \mathcal{M}^0 - k^3 \mathcal{M}^3 = k(\mathcal{M}^0 - \mathcal{M}^3) = 0$$

by Ward identity

$$\Rightarrow \mathcal{M}^0 = \mathcal{M}^3$$

Sums over polarizations should only involve physical pol. states $\varepsilon_1$ and $\varepsilon_2$.

$$\sum_{i=1,2} |\mathcal{M}|^2 = \sum_{i=1,2} |\varepsilon_i^\mu \mathcal{M}_\mu|^2 = |\mathcal{M}^1|^2 + |\mathcal{M}^2|^2$$

$$= |\mathcal{M}^1|^2 + |\mathcal{M}^2|^2 + |\mathcal{M}^3|^2 - |\mathcal{M}^0|^2$$

$$= (-g_{\mu\nu}) \mathcal{M}^\mu \mathcal{M}^{\nu *}$$

So can replace $\varepsilon_\mu \varepsilon_\nu^* \to -g_{\mu\nu}$. Unphysical polarization amplitudes cancel out.

Ward identity is useful check on matrix elements:

Compton scattering:

$$i\mathcal{M} = -ie^2 \varepsilon_\mu(k) \varepsilon_\nu^*(k') \bar{u}(p') \left[ \frac{\gamma^\nu \not{k} \gamma^\mu \overset{+2\gamma^\nu p^\mu}{}}{2p\cdot k} + \frac{\gamma^\mu \not{k}' \gamma^\nu - 2\gamma^\mu p^\nu}{2p\cdot k'} \right] u(p)$$

replace $\varepsilon_\mu(k) \to k_\mu$ $\qquad k = k' + p' - p$

$$i\mathcal{M} \to -i\varepsilon_\nu^*(k') \bar{u}(p') \left[ \frac{\gamma^\nu k^2 + 2\gamma^\nu k\cdot p}{2k\cdot p} + \frac{\not{k}\not{k}'\gamma^\nu - 2\not{k} p^\nu}{2p\cdot k'} \right] u(p) = 0$$