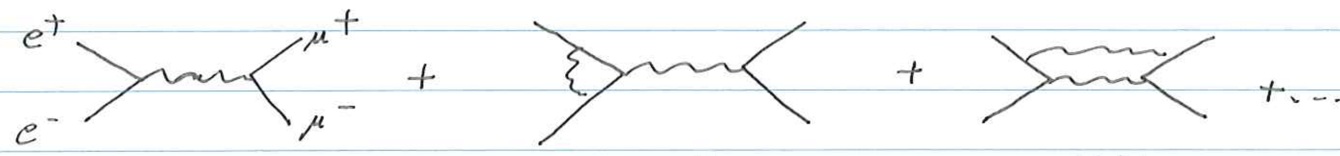
Radiative corrections

So far we have only consider the leading tree-level processes. However, also have terms higher order in perturbation theory

$e^+e^- \to \mu^+\mu^-$

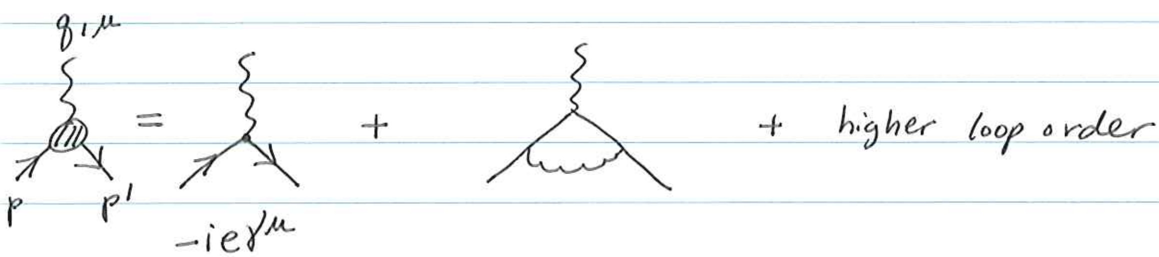
$i\mathcal{M} =$ 

$\mathcal{O}(e^2)$ tree-level $\mathcal{O}(e^4)$ $\mathcal{O}(e^4)$ one-loop

$\mathcal{O}(e^6)$ two-loop

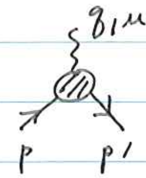
$$i\mathcal{M} \propto (e^2)^{L+1} \qquad L = \#\text{ of loops.}$$

Electron vertex corrections



$q = p' - p$

General matrix element:

$i\mathcal{M} =$ 

$= -ie\,\bar{u}(p')\,\Gamma^\mu\,u(p)$

at tree-level $\Gamma^\mu = \gamma^\mu$

most general form for Γ^μ:

$$\Gamma^\mu = A\gamma^\mu + (p'+p)^\mu B + (p'-p)^\mu C$$

any extra Dirac matrices must be contracted into $\not{p}$ or $\not{p}'$. Then can use $\not{p}\,u(p) = m\,u(p)$ and $\bar{u}(p')\not{p}' = m\bar{u}(p')$ to absorb these terms into A, B, C terms.

Note: Γ^μ must satisfy Ward identity $(q = p'-p)$

$$q^\mu\Gamma_\mu = A\not{q} + B\,(p'+p)\cdot(p'-p) + C\,(p'-p)^2 = 0$$

$$\bar{u}(p')\,\not{q}\,u(p) = \bar{u}(p')\,(m-m)\,u(p) = 0$$

$$q^\mu\Gamma_\mu = Cq^2 = 0 \quad \Rightarrow \quad C = 0$$

(note: $q^2 \neq 0$ in general; γ need not be external γ)

Use Gordon identity:

$$\bar{u}(p')\,(p'^\mu + p^\mu)\,u(p) = 2m\,\bar{u}(p')\,\gamma^\mu\,u(p) - \bar{u}(p')\,i\sigma^{\mu\nu}q_\nu\,u(p)$$

Most general form: $\Gamma^\mu(p,p') = F_1(q^2)\,\gamma^\mu + \dfrac{i\sigma^{\mu\nu}q_\nu}{2m}F_2(q^2)$

$F_{1,2}$ called <u>form factors</u>. Lorentz scalars $\rightarrow$ depend on p, p' through $p^2 = p'^2 = m^2$ or $p'\cdot p = -\frac{q^2}{2} + m^2$.

Physical interpretation:

Hamiltonian for non-relativistic e^- in external EM field $A^\mu = (\phi, \underline{A})$:

$$H = \frac{1}{2m} |\underline{p} - e\underline{A}|^2 + e\phi - \underline{\mu}_e \cdot \underline{B}$$

$$\underline{\mu}_e = -\frac{ge}{2m}\underline{S} = -\frac{ge}{4m}\underline{\sigma}$$

where g = gyromagnetic ratio

$$H = \frac{1}{2m}\left(|\underline{p}|^2 - 2e\,\underline{p}\cdot\underline{A} + e^2|\underline{A}|^2\right) + e\phi + \frac{ge}{4m}\underline{\sigma}\cdot\underline{B}$$

QFT calculation: couple e^- to classical external field A_μ^{cl}

$$\mathcal{L} = \mathcal{L}_{QED} - e\bar{\Psi}\gamma^\mu\Psi A_\mu^{cl}$$

Tree-level matrix element:

$$\langle p'|iT|p\rangle = \langle p'|(-ie)\int d^4x\, A_\mu^{cl}(x)\, \bar{\Psi}\gamma^\mu\Psi\,|p\rangle$$

$$= -ie\int d^4x\, A_\mu^{cl}\, e^{-i(p-p')\cdot x}\; \bar{u}(p')\gamma^\mu u(p)$$

If $A_\mu^{cl}(x)$ is a static field, then $A_\mu^{cl}(x) = A_\mu^{cl}(\underline{x})$

$$\Rightarrow \int d^4x\, e^{+iq\cdot x} A_\mu^{cl}(\underline{x}) = (2\pi)\,\delta(q^0)\,\tilde{A}_\mu^{cl}(\underline{q})$$

$$= (2\pi)\,\delta(q^0)\,(\tilde{\phi}(\underline{q}),\ \tilde{A}_i(\underline{q}))$$

$$\langle p' | iT | p \rangle = \underbrace{-ie\,\bar{u}(p')\gamma^\mu u(p)\,\tilde{A}^{cl}_\mu(q)}_{i\mathcal{M}} \cdot (2\pi)\delta(q^0)$$

Higher-order QED corrections:

$$i\mathcal{M} = -ie\,\bar{u}(p')\gamma^\mu u(p)\,\tilde{A}^{cl}_\mu(q) \longrightarrow -ie\,\bar{u}(p')\Gamma^\mu(p,p')u(p)\,\tilde{A}^{cl}_\mu(q)$$

Note: expect $i\mathcal{M} \sim -i\langle p' | H_{int} | p\rangle$ in momentum space.
$\sim -i\,(2m)\,\xi^\dagger_{s'}\,\tilde{V}(q)\,\xi_s$

(i) static scalar potential $A^{cl}_\mu(x) = (\phi(\underline{x}), 0, 0, 0)$

$$i\mathcal{M} = -ie\,\bar{u}(p')\Gamma^0 u(p)\,\phi(q)$$

consider $q \to 0$ limit (slowly varying potential, $|q|/m \ll 1$)

$$i\mathcal{M} \approx -ie\,\bar{u}(p)\gamma^0 F_1(0)\,u(p)\tilde{\phi} = -ie\,\underbrace{u^\dagger(p)u(p)}_{2m\,\xi^\dagger_{s'}\xi_s}\,F_1(0)\,\tilde{\phi}(q)$$

$$\approx -ie\,2m\,\xi^\dagger_{s'}\xi_s\,F_1(0)\,\tilde{\phi}(q)$$

$\Rightarrow$ potential is $\tilde{V}(q) = e F_1(0)\,\tilde{\phi}(q)$

$$V(\underline{x}) = e F_1(0)\,\phi(\underline{x})$$

$F_1(0)$ is electric charge of e^- (in units of $-e$)
Since at tree-level $F_1(0) = F_1(q^2) = 1$, then $F_1(0) = 1$ at any order in perturbation theory.
(Loop corrections to $F_1(q^2)$ must vanish as $q^2 \to 0$.)

(ii) static vector potential: $A_\mu^{cl}(x) = (0, A_i^{cl}(\underline{x}))$

$$i\mathcal{M} = +ie\,\bar{u}(p')\left[F_1 \gamma^i + \frac{i\sigma^{i\nu} q_\nu}{2m} F_2\right] u(p)\, \tilde{A}^i(\cancel{q})$$

Non relativistic limit: $\sqrt{p\cdot\sigma} = \sqrt{m - \underline{p}\cdot\underline{\sigma}} \approx \sqrt{m}\left(1 - \frac{\underline{p}\cdot\underline{\sigma}}{2m}\right)$

$\sqrt{p\cdot\bar{\sigma}} \approx \sqrt{m}\left(1 + \frac{\underline{p}\cdot\underline{\sigma}}{2m}\right)$

F_1 term:

$$\bar{u}(p')\gamma^i u(p) = m\left(\xi_{s'}^\dagger\left(1 + \frac{\underline{p}'\cdot\underline{\sigma}}{2m}\right), \xi_{s'}^\dagger\left(1 - \frac{\underline{p}'\cdot\underline{\sigma}}{2m}\right)\right)\begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}\begin{pmatrix} \left(1 - \frac{\underline{p}\cdot\underline{\sigma}}{2m}\right)\xi_s \\ \left(1 + \frac{\underline{p}\cdot\underline{\sigma}}{2m}\right)\xi_s \end{pmatrix}$$

$$= 2m\,\xi_{s'}^\dagger\left(\frac{\underline{p}'\cdot\underline{\sigma}}{2m}\sigma^i + \sigma^i\frac{\underline{p}\cdot\underline{\sigma}}{2m}\right)\xi_s$$

use identity $\sigma^i\sigma^j = \delta^{ij} + i\varepsilon^{ijk}\sigma^k$ and keep only terms leading in $q \to 0$ limit.

$$\bar{u}(p')\gamma^i u(p) \approx 2m\,\xi_{s'}^\dagger\left(\frac{2p^i}{2m} - \frac{i\varepsilon^{ijk} q^j \sigma^k}{2m}\right)\xi_s$$

F_2 term:

$$\bar{u}(p')\frac{i\sigma^{i\nu} q_\nu}{2m} u(p) \approx -\bar{u}(p')\frac{i\sigma^{ij} q^j}{2m} u(p)$$

$$\approx -2m\,\xi_{s'}^\dagger\frac{i\varepsilon^{ijk} q^j \sigma^k}{2m}\xi_s$$

using $u(p) \approx \sqrt{m}\begin{pmatrix}\xi_s \\ \xi_s\end{pmatrix}$

$\sigma^{ij} = \frac{i}{2}[\gamma^i, \gamma^j] = \frac{1}{2}\varepsilon^{ijk}\sigma^k$

So we have:

$$i\mathcal{M} = i\,(2m)\,\xi_{s'}^{\dagger}\left(-e\,\frac{2p\cdot\tilde{A}(q)}{2m}\,F_1(0) + e\,i\,\frac{\varepsilon^{ijk}\tilde{A}^i(q)\,q^j\sigma^k}{2m}\,(F_1+F_2)\right)\xi_s$$

Note: $\underline{B}(\underline{x}) = \underline{\nabla}\times\underline{A}$

$$\tilde{B}^i(q) = \int d^3x\, e^{-iq\cdot x}(\underline{\nabla}\times\underline{A})^i = \int d^3x\, e^{-iq\cdot x}\,\varepsilon^{ijk}\partial_j A_k$$

$$= -\int d^3x\; i q^j A^k \varepsilon^{ijk} = -i\,\varepsilon^{ijk} q^j \tilde{A}(q)^k$$

$$i\mathcal{M} = -i\,(2m)\,\xi_{s'}^{\dagger}\left(-eF_1\frac{2p\cdot\tilde{A}(q)}{2m} \;-\; (F_1+F_2)\,e\,\frac{\underline{B}\cdot\underline{\sigma}}{2m}\right)\xi_s$$

$$\frac{1}{2m}\left(-2e\,p\cdot\underline{A}\right) \qquad\qquad \frac{ge}{4m} = \frac{(F_1+F_2)\,e}{2m}$$

$$\Rightarrow\quad g = 2\left(F_1(0)+F_2(0)\right) = 2 + 2F_2(0)$$

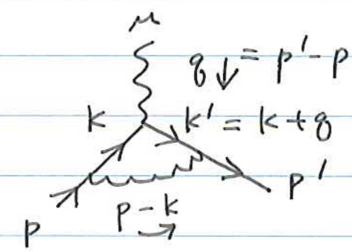
at tree-level: $F_2(0) = 0$. $\uparrow$ standard prediction from Dirac theory $g=2$

$F_2(0) = \mathcal{O}(\alpha)$ is prediction from higher-order corrections in QED

Evaluating electron vertex correction

$$-ie\,\bar{u}(p')\,\Gamma^{\mu}\,u(p) = -ie\,\bar{u}(p')\,(\underbrace{\gamma^{\mu}}_{\text{tree}} + \underbrace{\delta\Gamma^{\mu}}_{\text{1-loop}})\,u(p)$$

$$i\mathcal{M} = -ie\,\bar{u}(p')\,\delta\Gamma^{\mu}\,u(p) =$$



$$= \int \frac{d^4k}{(2\pi)^4}\, \bar{u}(p') \overset{(-ie\gamma^{\nu})}{\vee} \frac{i(\not{k}'+m)}{k'^2-m^2+i\varepsilon}(-ie\gamma^{\mu})\frac{i(\not{k}+m)}{k^2-m^2+i\varepsilon}(-ie\gamma_{\nu})\,u(p)$$

$$\times \quad \frac{-i}{(p-k)^2+i\varepsilon}$$

$$= -e^3 \int \frac{d^4k}{(2\pi)^4} \frac{\bar{u}(p')\gamma^{\nu}(\not{k}'+m)\gamma^{\mu}(\not{k}+m)\gamma_{\nu}u(p)}{(k^2-m^2+i\varepsilon)(k'^2-m^2+i\varepsilon)((p-k)^2+i\varepsilon)}$$

$$= +2e^3 \int \frac{d^4k}{(2\pi)^4} \frac{\bar{u}(p')(\not{k}\gamma^{\mu}\not{k}' - 2m(k+k')^{\mu} + m^2\gamma^{\mu})u(p)}{(k^2-m^2+i\varepsilon)(k'^2-m^2+i\varepsilon)((p-k)^2+i\varepsilon)}$$

<u>Feynman parameters</u>: trick for evaluating integral

$$\frac{1}{AB} = \int_0^1 dx \frac{1}{(Ax+B(1-x))^2} = \int_0^1 dx\,dy\,\delta(x+y-1)\frac{1}{(Ax+By)^2}$$

general formula:

$$\frac{1}{A_1A_2\cdots A_n} = \int_0^1 dx_1\,dx_2\cdots dx_n\,\delta(x_1+\cdots+x_n-1)\frac{(n-1)!}{(A_1x_1\cdots+A_nx_n)^n}$$

$x_1,\ldots x_n$ are Feynman parameters.

$n=3$ case: $A_1 = k^2 - m^2 + i\varepsilon$, $A_2 = k'^2 - m^2 + i\varepsilon$, $A_3 = (p-k)^2 + i\varepsilon$

$$\frac{1}{A_1 A_2 A_3} = 2\int_0^1 dx\,dy\,dz\;\delta(x+y+z-1)\,\frac{1}{(A_1 x + A_2 y + A_3 z)^3}$$

$$\text{denominator} = A_1 x + A_2 y + A_3 z$$

$$= (k^2 - m^2)x + (k'^2 - m^2)y + ((p-k)^2)z + i\varepsilon\,\underbrace{(x+y+z)}_{1}$$

$$= k^2 x + (k^2 + 2k\cdot q + q^2)y + (k^2 - 2k\cdot p + p^2)z$$
$$- m^2 x - m^2 y + i\varepsilon$$

$$= k^2 + 2k\cdot(qy - pz) + q^2 y + p^2 z - m^2(x+y) + i\varepsilon$$

$$= (k + yq - zp)^2 - (yq - zp)^2 + q^2 y + p^2 z$$
$$- m^2(x+y) + i\varepsilon$$

$$= \ell^2 - M^2 + i\varepsilon$$

where $\ell = k + yq - zp$

$$M^2 = m^2(1-z) + y^2 q^2 + z^2 p^2 - 2zy\,p\cdot q - q^2 y - p^2 z$$

note: $p\cdot q = p\cdot(p' - p) = p\cdot p' - m^2$ $\cancel{= -\frac{1}{2}(p'-p)^2}$

$q^2 = 2m^2 - 2p\cdot p' \Rightarrow p\cdot p' = m^2 - q^2/2$

$p\cdot q = -q^2/2$

$$M^2 = m^2(1-z) + y^2 q^2 + z^2 m^2 + 2zy\,\frac{q^2}{2} - q^2 y - m^2 z$$

$$= m^2(1-z)^2 + q^2 y(z + y - 1) = m^2(1-z)^2 - xy\,q^2$$

Now perform shift under the integral : $k \to l$

$$\int \frac{d^4k}{(2\pi)^4} = \int \frac{d^4 l}{(2\pi)^4} \quad , \quad k = l - yq + zp$$

$$i\mathcal{M} = 2e^3 \int \frac{d^4 l}{(2\pi)^4} \overset{2\int dx dy dz}{\vee} \bar{u}(p')\Big[(\not{l} - y\not{q} + z\not{p})\gamma^\mu(\not{l} - y\not{q} + z\not{p} + \not{q}) + m^2\gamma^\mu - 2m\big(2(l - yq + zp) + q\big)^\mu\Big]u(p) \frac{1}{(l^2 - M^2 + i\varepsilon)^3}$$

terms odd in l vanish by $l \to -l$ antisymmetry

$$i\mathcal{M} = 2e^3 \int \frac{d^4 l}{(2\pi)^4} \overset{2\int dx dy dz}{\vee} \bar{u}(p')\Big[\not{l}\gamma^\mu\not{l} + (-y\not{q} + z\not{p})\gamma^\mu((1-y)\not{q} + z\not{p}) + m^2\gamma^\mu - 2m\big((1-2y)q + 2zp\big)^\mu\Big]u(p) \frac{1}{(l^2 - M^2 + i\varepsilon)}$$

$$= 2e^3 \cdot 2\int dx\, dy dz\, \delta(x+y+z-1)$$

$$\times \bar{u}(p)\Big\{ I_{\alpha\beta}\gamma^\alpha\gamma^\mu\gamma^\beta + I_0\Big[(-y\not{q} + z\not{p})\gamma^\mu((1-y)\not{q} + z\not{p}) + m^2\gamma^\mu - 2m((1-2y)q + 2zp)^\mu\Big]\Big\}u(p)$$

$$I_0 = \int \frac{d^4 l}{(2\pi)^4} \frac{1}{(l^2 - M^2 + i\varepsilon)^3} \qquad I_{\alpha\beta} = \int \frac{d^4 l}{(2\pi)^4} \frac{l_\alpha l_\beta}{(l^2 - M^2 + i\varepsilon)^3}$$

note: $I_{\alpha\beta}$ must vanish if $\alpha \neq \beta$ by antisymmetry.

$$\Rightarrow \quad I_{\alpha\beta} \propto g_{\alpha\beta} \quad \Rightarrow \quad I_{\alpha\beta} = \frac{1}{4}\int \frac{d^4 l}{(2\pi)^4} \frac{l^2 g_{\alpha\beta}}{(l^2 - M^2 + i\varepsilon)^3}$$

check: $\quad g^{\alpha\beta} I_{\alpha\beta} = \int \frac{d^4 l}{(2\pi)^4} \frac{l^2}{(l^2 - M^2 + i\varepsilon)^3} \equiv I_2$

$$i\mathcal{M} = 2e^3 \cdot 2\int_0^1 dx\,dy\,dz\ \delta(x+y+z-1)\ x\ \bar{u}(p')\Big\{-\tfrac{1}{2}\gamma^\mu I_2$$

$$+ I_0\big[(-y\not{q}+z\not{p})\gamma^\mu((1-y)\not{q}+z\not{p}) + m^2\gamma^\mu - 2m((1-2y)q^\mu + 2zp^\mu)\big]\Big\}u(p)$$

Simplify I_0 term using Dirac equation: $\not{p}\,u(p) = m\,u(p)$

$\bar{u}(p')\not{p}' = m\,\bar{u}(p')$

and also $q = p' - p$ and $x+y+z=1$.

Get result:

$$i\mathcal{M} = 2e^3 \cdot 2\int_0^1 dx\,dy\,dz\ \delta(x+y+z-1)$$

$$\times\ \bar{u}(p')\Big\{\Big(-\tfrac{1}{2} I_2 + \big[(1-x)(1-y)q^2 + (1-2z-z^2)m^2\big] I_0\Big)\gamma^\mu \quad \leftarrow \text{A term}$$

$$+ (p^\mu + p'^\mu)\, m\, z(z-1)\, I_0 \quad \leftarrow \text{B term}$$

$$- q^\mu\, m\,(x+y+1)(x-y)\, I_0\Big\}\, u(p) \quad \leftarrow \text{C term}$$

C term vanishes by antisymmetry under $x \leftrightarrow y$

Use Gordon identity: $(p^\mu + p'^\mu) \rightarrow 2m\gamma^\mu - i\sigma^{\mu\nu}q_\nu$

$$i\mathcal{M} = 2e^3 \cdot 2\int_0^1 dx\,dy\,dz\ \delta(x+y+z-1)$$

$$\times\ \bar{u}(p')\Big\{\Big(-\tfrac{1}{2} I_2 + (1-x)(1-y)q^2 I_0 + (1-4z+z^2)m^2 I_0\Big)\gamma^\mu$$

$$+ I_0\,\frac{i\sigma^{\mu\nu}q_\nu}{2m}\big(2m^2 z(1-z)\big)\Big\}\, u(p)$$

$$= -ie\,\bar{u}(p')\Big(\underset{\uparrow}{\delta F_1}\gamma^\mu + F_2\,\frac{i\sigma^{\mu\nu}q_\nu}{2m}\Big)u(p)$$

1-loop contribution: $F_1 = 1 + \delta F_1$

So the form factors are:

$$\delta F_1(q^2) = 2ie^2 \cdot 2\int_0^1 dx\,dy\,dz\,\delta(x+y+z-1)$$

$$\times \left\{ -\tfrac{1}{2}\,\mathcal{I}_2 + (1-x)(1-y)\,q^2\,\mathcal{I}_0 + m^2(1-4z+z^2)\,\mathcal{I}_0 \right\}$$

$$F_2(q^2) = 2ie^2 \cdot 2\int_0^1 dx\,dy\,dz\,\delta(x+y+z-1)$$

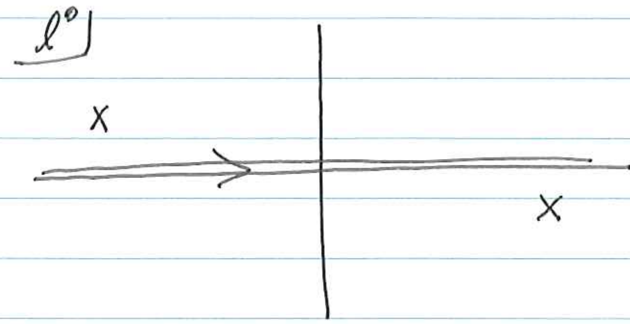
$$\times\ \mathcal{I}_0 \cdot 2m^2 z(1-z)$$

<u>Evaluate F_2:</u> need to compute $\mathcal{I}_0$:

$$\mathcal{I}_0 = \int \frac{d^4 l}{(2\pi)^4} \frac{1}{(l^2 - M^2 + i\varepsilon)^3}$$

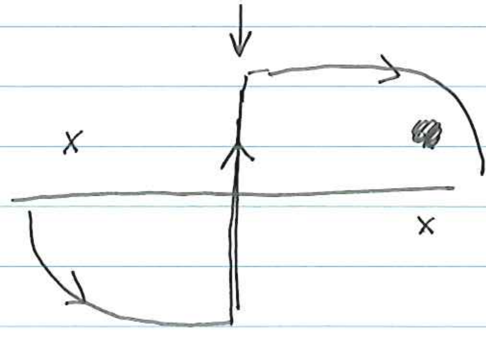
$$= \int \frac{d^3 l}{(2\pi)^3} \int_{-\infty}^{\infty} \frac{dl^0}{2\pi} \frac{1}{(l_0^2 - (|\underline{l}|^2 + M^2) + i\varepsilon)^3}$$

Do $\int dl^0$ integral by contour integration



poles at

$$l^0 = \pm\sqrt{|\underline{l}|^2 + M^2 - i\varepsilon}$$
$$= \pm\sqrt{|\underline{l}|^2 + M^2} \mp i\varepsilon$$



deform contour without passing through poles

quarter-circles contributions $\to 0$

$$\Rightarrow \int_{-\infty}^{\infty} dl^0 = \int_{-i\infty}^{i\infty} dl^0$$

Define $\underline{\text{Euclidean}}$ momentum l_E^μ

such that $\underline{l}_E = \underline{l}$ and $l^0 = i\, l_E^0$

Thus $\displaystyle\int_{-\infty}^{\infty} dl^0 = \int_{-i\infty}^{i\infty} dl^0 = i \int_{-\infty}^{\infty} dl_E^0$

So $\displaystyle\int \frac{d^4 l}{(2\pi)^4} \frac{1}{(l^2 - M^2 + i\varepsilon)^3} = i \int \frac{d^4 l_E}{(2\pi)^4} \frac{1}{(-l_E^2 - M^2)^3}$ now can drop ε

$\displaystyle = i \frac{-1}{(2\pi)^4} \int d\Omega_4 \int_0^\infty \frac{l_E^3\, dl_E}{(l_E^2 + M^2)^3}$ in 4-d spherical coords.

$\int d\Omega_4$ = surface area of unit sphere in 4-d.
$= 2\pi^2$

$\displaystyle\int_0^\infty \frac{l_E^3\, dl_E}{(l_E^2 + M^2)^3} = \frac{1}{4M^2}$

So $\displaystyle I_0 = -\frac{i}{(2\pi)^4} \cdot 2\pi^2 \frac{1}{4M^2} = -\frac{i}{32\pi^2 M^2}$

$$F_2(q^2) = \frac{4e^2}{32\pi^2} \int_0^1 dx\, dy\, dz\, \delta(x+y+z-1) \frac{2m^2 z(1-z)}{M^2(1-z)^2 - xyq^2}$$

$$= \frac{\alpha}{\pi} \int_0^1 dx\, dy\, dz\, \delta(x+y+z-1) \left(\frac{m^2 z(1-z)}{m^2(1-z)^2 - xy\, q^2} \right)$$

$$F_2(0) = \frac{\alpha}{\pi} \int_0^1 dx\, dy\, dz\, \delta(x+y+z-1) \left(\frac{z}{1-z} \right)$$

$$= \frac{\alpha}{\pi} \int_0^1 dz \int_0^{1-z} dx \left(\frac{z}{1-z} \right) = \frac{\alpha}{\pi} \int_0^1 dz\, z = \frac{\alpha}{2\pi}$$

Correction to $g=2$ due to QED loop corrections.

anomalous magnetic moment

$$a_e = \frac{g-2}{2} = F_2(0) = \frac{\alpha}{2\pi} \quad \text{at 1-loop (Schwinger)}$$

$$\approx 0.0011614$$

$$a_e^{obs} \doteq 0.00115965218073(28)$$

agrees at $\mathcal{O}(10^{-3}$ level$)$ $\rightarrow$ 2-loop effects

Evaluate F_1: $\quad F_1(q^2) = 1 + \delta F_1(q^2)$

$$I_2 = \int \frac{d^4 \ell}{(2\pi)^4} \frac{\ell^2}{(\ell^2 - M^2 + i\varepsilon)^3} = i \int \frac{d^4 \ell_E}{(2\pi)^4} \frac{-\ell_E^2}{(-\ell_E^2 - M^2)^3}$$

$$= i \frac{2\pi^2}{(2\pi)^4} \underbrace{\int_0^\infty \frac{\ell_E^5 \, d\ell_E}{(\ell_E^2 + M^2)^3}}_{\sim \int^\infty \frac{d\ell_E}{\ell_E} \sim \log \infty}$$

Integral is divergent.

Divergences are a ubiquitous feature of loop corrections in QFT. Naively, breaks perturbative expansion:

$$F_1(q^2) = \underbrace{1}_{\text{tree}} + \underbrace{\frac{\alpha}{4\pi} \times \mathcal{O}(\infty)}_{\text{"small" loop correction.}}$$

Nevertheless, divergences can be removed in a systematic way through renormalization (beyond scope of this course).

First step of removing divergences is regularization. Introduce fictitious parameter to regulate the divergence, and then take the limit that the parameter "goes away"

Pauli-Villars regularization

Prescription: add new term to photon propagator

$$\frac{1}{(k-p)^2+i\varepsilon} \longrightarrow \frac{1}{(k-p)^2+i\varepsilon} - \frac{1}{(k-p)^2-\Lambda^2+i\varepsilon}$$

where Λ is a large mass for a fictitious new photon whose contribution cancels the usual photon propagator.

Limit $\Lambda \to \infty \Rightarrow$ extra term $\to 0$

But for $\Lambda \neq \infty$, extra term cuts off integral where $k^2 \gtrsim \Lambda^2$.

Then we have:

$$I_2 = \int \frac{d^4 l}{(2\pi)^4} \left\{ \frac{l^2}{(l^2 - M^2 + i\varepsilon)} - \frac{l^2}{(l^2 - M_\Lambda^2 + i\varepsilon)^3} \right\}$$

where $M_\Lambda^2 = m^2(1-z)^2 - xyq^2 + z\Lambda^2$

$$I_2 = \frac{2\pi^2 i}{(2\pi)^4} \int_0^\infty dl_E \, l_E^5 \left(\frac{1}{(l_E^2+M^2)^3} - \frac{1}{(l_E^2+M_\Lambda^2)^3} \right)$$

$$= \frac{i}{(4\pi)^2} \log\left(M_\Lambda^2 / M^2 \right) \sim \log \Lambda \quad \text{for } \Lambda \to \infty.$$

Neglect $\mathcal{O}(1/\Lambda^2)$ corrections to I_0

$$F_1(q^2) = 1 + 4ie^2 \int_0^1 dx\,dy\,dz\,\delta(x+y+z-1)$$
$$\times \left\{ -\frac{1}{2}\frac{i}{(4\pi)^2}\log\left(M_\Lambda^2/M^2\right) + \left[(1-x)(1-y)q^2 + (1-4z+z^2)m^2\right]\frac{-i}{2(4\pi)^2 M^2} \right\}$$

$$= 1 + \frac{\alpha}{2\pi}\int_0^1 dx\,dy\,dz\,\delta(x+y+z-1)$$
$$\times \left\{ \log\left(M_\Lambda^2/M^2\right) + \frac{(1-x)(1-y)q^2 + m^2(1-4z+z^2)}{M^2} \right\}$$

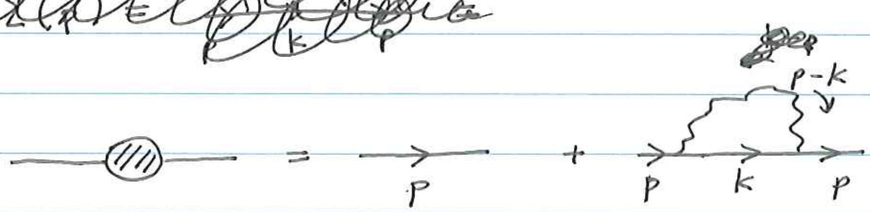
$$= 1 + \frac{\alpha}{2\pi}\int_0^1 dx\,dy\,dz\,\delta(x+y+z-1)$$
$$\times \left\{ \log\left(\frac{m^2(1-z)^2 - xy\,q^2 + z\Lambda^2}{m^2(1-z)^2 - xy\,q^2}\right) + \frac{(1-x)(1-y)q^2 + m^2(1-4z+z^2)}{m^2(1-z)^2 - xy\,q^2} \right\}$$

Previously we argued $F_1(0) = 1$, but we have:

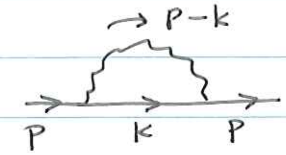
$$F_1(0) = 1 + \frac{\alpha}{2\pi}\int_0^1 dx\,dy\,dz\,\delta(x+y+z-1)$$
$$\times \left\{ \log\left(\frac{m^2(1-z)^2 + z\Lambda^2}{m^2(1-z)^2}\right) + \frac{m^2(1-4z+z^2)}{m^2(1-z)^2} \right\}$$

External leg corrections

Photon corrections to electron propagator



$$= \frac{i(\not p + m)}{p^2 - m^2} = \frac{i(\not p + m)}{(\not p + m)(\not p - m)} = \frac{i}{\not p - m}$$



$$= \frac{i}{\not p - m} \int \frac{d^4k}{(2\pi)^4} (-i\gamma^\mu e) \frac{i(\not k + m)}{k^2 - m^2 + i\varepsilon} (-ie\gamma_\mu) \frac{-i}{(p-k)^2 - i\varepsilon} \frac{i}{\not p - m}$$

$$= \frac{i}{\not p - m} \left(-i\Sigma(\not p)\right) \frac{i}{\not p - m}$$

where $-i\Sigma(\not p) = -\int \frac{d^4k}{(2\pi)^4} \frac{e^2 \gamma^\mu (\not k + m)\gamma_\mu}{(k^2 - m^2 + i\varepsilon)((k-p)^2 - i\varepsilon)}$

$$= 2e^2 \int \frac{d^4k}{(2\pi)^4} \frac{\not k - 2m}{(k^2 - m^2 + i\varepsilon)((k-p)^2 - i\varepsilon)}$$

use Feynman parameters & shifting momentum variable:

$$-i\Sigma(\not p) = 2e^2 \int_0^1 dx \int \frac{d^4\ell}{(2\pi)^4} \frac{x\not p - 2m}{(\ell^2 - M^2 + i\varepsilon)^2}$$

where $\ell = k - xp$, $M^2 = (1-x)m^2 - x(1-x)p^2$

This integral is also log-divergent. Introduce Pauli-Villars regulator:

$$\frac{1}{(k-p)^2 + i\varepsilon} \longrightarrow \frac{1}{(k-p)^2 + i\varepsilon} - \frac{1}{(k-p)^2 - \Lambda^2 + i\varepsilon}$$

Then $\int \frac{d^4\ell}{(2\pi)^4} \frac{1}{(\ell^2 - M^2 + i\varepsilon)^2} \longrightarrow \int \frac{d^4\ell}{(2\pi)^4} \left(\frac{1}{(\ell^2 - M^2 + i\varepsilon)^2} - \frac{1}{(\ell^2 - M_\Lambda^2 + i\varepsilon)^2} \right)$

$$= \frac{i}{(4\pi)^2} \log\left(M_\Lambda^2 / M^2\right)$$

$$\Sigma(\not p) = \frac{\alpha}{2\pi} \int_0^1 dx\, (2m - x\not p) \log\left(M_\Lambda^2 / M^2\right)$$

where $M_\Lambda^2 = (1-x)m^2 - x(1-x)p^2 + x\Lambda^2 \approx x\Lambda^2$
for $\Lambda^2 \gg m^2, p^2$.

Consider an infinite series:

$$= \frac{i}{\not p - m} + \frac{i}{\not p - m}(-i\Sigma)\frac{i}{\not p - m} + \frac{i}{\not p - m}(-i\Sigma)\frac{i}{\not p - m}(-i\Sigma)\frac{i}{\not p - m} + \cdots$$

$$= \frac{i}{\not p - m} \cdot \sum_{n=0}^{\infty} \left(\frac{\Sigma(\not p)}{\not p - m}\right)^n = \frac{i}{\not p - m} \frac{1}{1 - \left(\frac{\Sigma}{\not p - m}\right)}$$

$$= \frac{i}{\not p - m - \Sigma(\not p)} \approx \frac{i}{\not p - m - \Sigma(m) - \frac{\partial \Sigma}{\partial \not p}\Big|_{\not p = m}(\not p - m)}$$

$$\approx \frac{i}{\not p - m - \Sigma(m)} \frac{1}{1 - \frac{\partial \Sigma}{\partial \not p}\Big|_{\not p = m}} \approx \frac{i Z_2}{\not p - m - \Sigma(m)} = \frac{i Z_2}{\not p - m_*}$$

where $Z_2 = 1 + \frac{\partial \Sigma}{\partial \not p}\Big|_{\not p = m}$

pole in propagator is no longer at $\not p = m$ (or $p^2 = m^2$)
but at $\not p = m_* = m + \delta m$

where $\delta m = \Sigma(m) = \frac{\alpha m}{2\pi} \int_0^1 dx\, (2 - x) \log\left(\frac{x \Lambda^2}{(1-x)^2 m^2}\right)$

$$\rightarrow \frac{3\alpha}{4\pi} m \log\left(\Lambda^2 / m^2\right) \quad \text{in } \Lambda \to \infty \text{ limit.}$$

physical electron mass is m_* = finite mass.

m = parameter in Lagrangian, not physical mass
→ "bare mass"
→ must be infinite to absorb infinite 1-loop correction.

$$\left.\frac{\partial \Sigma}{\partial \not p}\right|_{\not p = m} = \frac{\alpha}{2\pi}\int_0^1 dx \left[-x \log\left(\frac{x\Lambda^2}{m^2(1-x)^2}\right) + \frac{4x-2x^2}{1-x}\right]$$

$$= \frac{\alpha}{2\pi}\int_0^1 dx \left[x \log\left(\frac{x\Lambda^2}{m^2(1-x)^2}\right) + \frac{2x(2-x)}{1-x}\right]$$

LSZ formula: external states are modified by interactions (free states vs. interacting states)

physical mass shift: $m \to m + \delta m$

residue: $|p\rangle_0 \to \sqrt{Z}\,|p\rangle_0 = |p\rangle$

External leg corrections incuded by multiplying by $\sqrt{Z} = \sqrt{1 + \left.\frac{\partial \Sigma}{\partial \not p}\right|_{\not p = m}} \approx 1 + \frac{1}{2}\left.\frac{\partial \Sigma}{\partial \not p}\right|_{\not p = m}$ for each leg

(here just considering electron legs)

Should write: $\Gamma^\mu \to Z\Gamma^\mu = Z(\gamma^\mu + \delta\Gamma^\mu)$

$$= \gamma^\mu + \left.\frac{\partial \Sigma}{\partial \not p}\right|_{\not p = m}\gamma^\mu + \delta\Gamma^\mu$$

So total F_1 is:

$$F_1(q^2) = 1 + \delta F_1(q^2) + \left.\frac{\partial \Sigma}{\partial \not p}\right|_{\not p = m}$$

tree + vertex + leg.

Consider setting $q^2=0$:

$$\delta F_1(q^2=0) + \frac{\partial\Sigma}{\partial \not{p}}\Big|_{\not{p}=m} = \frac{\alpha}{2\pi}\int_0^1 dz \Big\{ (1-z)\log\left(\frac{z\Lambda^2}{m^2(1-z)^2}\right) + \frac{1-4z+z^2}{1-z}$$

$$- z\log\left(\frac{z\Lambda^2}{m^2(1-z)^2}\right) + \frac{2z(2-z)}{1-z}\Big\}$$

$$= \frac{\alpha}{2\pi}\int_0^1 dz \left\{ (1-2z)\log\left(\frac{z\Lambda^2}{m^2(1-z)^2}\right) + \frac{1-z^2}{1-z} \right\}$$

$$\cancel{= \frac{\alpha}{2\pi}\int_0^1 dz \{[(1-z)-z]+}$$

$$= \frac{\alpha}{2\pi}\int_0^1 dz \left\{ (1-2z)\log\left(\frac{\Lambda^2}{m^2}\right) + (1-2z)\log\left(\frac{z}{(1-z)^2}\right) + 1 + z \right\}$$

$$\int_0^1 dz\,(1-2z) = 0$$

$$\int_0^1 dz \left[(1-2z)\log\frac{z}{(1-z)^2} + 1 + z \right] = 0$$

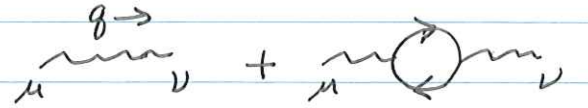
So $\delta F_1(0) = -\frac{\partial\Sigma}{\partial \not{p}}\Big|_{\not{p}=m}$

$\Rightarrow \quad F_1(q^2) = 1 + \delta F_1(q^2) - \delta F_1(0)$

manifestly $F_1(0) = 1$ at $q^2=0$.

Vacuum polarization

Photon propagator also gets loop corrections

$i\,\Pi_{\mu\nu}(q) =$  at 1-loop

Amplitude must satisfy Ward identity: $q_\mu \Pi^{\mu\nu} = 0$, $q_\nu \Pi^{\mu\nu} = 0$

Also, can only depend on $g^{\mu\nu}$ and $q^\mu q^\nu$ (on 2-index objects)

$$\Rightarrow \Pi_{\mu\nu} = (q^2 g_{\mu\nu} - q_\mu q_\nu)\underbrace{\Pi(q^2)}_{\text{function of } q^2}$$

1-loop corrections can be resummed (similar to electron)

$$= \frac{-i g_{\mu\nu}}{q^2} + \frac{-i g_{\mu\lambda}}{q^2}\, i\Pi^{\lambda\kappa}\, \frac{-i g_{\kappa\nu}}{q^2} + \dots$$

$$\cancel{= \frac{-i g_{\mu\nu}}{q^2} +}$$

Note: $(q^2 g_{\mu\nu} - q_\mu q_\nu) g^{\nu\lambda} (q^2 g_{\lambda\kappa} - q_\lambda q_\kappa)$

$$= q^4 g_{\mu\kappa} - 2q^2 q_\mu q_\kappa + q^2 q_\mu q_\kappa$$

$$= q^2 (q^2 g_{\mu\kappa} - q_\mu q_\kappa)$$

$$\cancel{= \frac{-i g_{\mu\nu}}{q^2} + \frac{-i}{q^2}(q^2 g_{\lambda\nu} - q_\mu q_\lambda)\Pi(q^2)\frac{1}{q^2}}$$

$$\cancel{\frac{i}{q^2} q^2 \Pi \frac{1}{q^4}}$$


$$= \frac{-ig_{\mu\nu}}{q^2} + \frac{-ig_{\mu\lambda}}{q^2}\frac{\Pi^{\lambda}{}_{\nu}}{q^2} + \frac{-ig_{\mu\lambda}}{q^2}\frac{\Pi^{\lambda}{}_{\kappa}\Pi^{\kappa}{}_{\nu}}{q^2 q^2} + \dots$$

$$= \frac{-ig_{\mu\nu}}{q^2} + \frac{-ig_{\mu\lambda}}{q^2}\frac{\Pi}{q^2}(q^2 g^{\lambda}{}_{\nu} - q^{\lambda} q_{\nu})$$

$$- \frac{ig_{\mu\lambda}}{q^2} \frac{\Pi^2 q^2 (q^2 g^{\lambda}{}_{\nu} - q^{\lambda} q_{\nu})}{q^4} + \dots$$

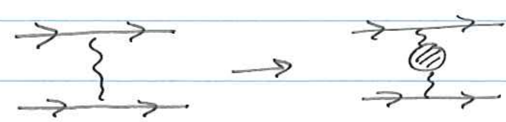
$$= -\frac{ig_{\mu\nu}}{q^2}(1 + \Pi + \Pi^2 + \dots) + \frac{i q_\mu q_\nu}{q^4}(\Pi + \Pi^2 + \dots)$$

$$= -\frac{i}{q^2(1-\Pi(q^2))}\left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right) - \frac{i}{q^4} q_\mu q_\nu$$

$q_\mu q_\nu$ terms vanish when photon legs are contracted with $\varepsilon^\mu(q)$ (external photon) or $\bar\Psi \gamma^\mu \Psi$ (fermion vertex)

$\Rightarrow$  $\dfrac{-ig_{\mu\nu}}{q^2(1-\Pi(q^2))}$

Coulomb interaction:
in low-q^2 limit:



$$\frac{e^2 g_{\mu\nu}}{q^2} \rightarrow \frac{Z_e\, e^2 g_{\mu\nu}}{q^2}$$

$$\text{where } Z_e = \frac{1}{1-\Pi(0)}$$

This is called charge renormalization:

relabel Lagrangian parameter $e \rightarrow e_0$ "bare charge"

$$\mathcal{L}_{int} = -e_0 \bar\Psi \gamma^\mu \Psi A_\mu$$

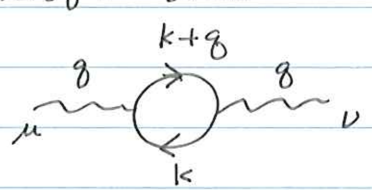
physical charge $= e = \sqrt{Z_e}\, e_0$

Can define effective α as function of q^2:

$$\alpha(q^2) = \frac{e^2}{4\pi} = \frac{e_0^2/4\pi}{1-\Pi(q^2)} = \frac{\alpha(0)\,(1-\Pi(0))}{1-\Pi(q^2)}$$

$$\simeq \frac{\alpha(0)}{1-(\Pi(q^2)-\Pi(0))}$$

Now compute $\Pi(q^2)$ from  :

$i\Pi_{\mu\nu}(q^2) =$ 

$$= -(-ie)^2 \int \frac{d^4k}{(2\pi)^4} \frac{\mathrm{Tr}\left[\gamma^\mu(\not{k}+m)\gamma^\nu(\not{k}+\not{q}+m)\right]}{(k^2-m^2+i\varepsilon)((k+q)^2-m^2+i\varepsilon)}$$

↑ note $e \approx e_0 + \mathcal{O}(\alpha)$

$$= -4e^2 \int \frac{d^4k}{(2\pi)^4} \frac{k^\mu(k+q)^\nu + k^\nu(k+q)^\mu - g^{\mu\nu}((k+q)\cdot k - m^2)}{(k^2-m^2+i\varepsilon)((k+q)^2-m^2+i\varepsilon)}$$

Use Feynman parameters & momentum shift $k \to l$.

Momentum integral takes form $\displaystyle\int \frac{d^4 l_E}{(2\pi)^4} \frac{l_E^2 + \dots}{(l_E^2+M^2)^2} \sim \int^\infty dl_E\, l_E \to \infty$

$\Rightarrow$ $\Pi(0)$ is divergent $\Rightarrow$ e_0 is divergent, but physical charge e is finite.

Another useful trick for regularizing divergences is <u>dimensional reduction</u>.

Note: in d-dimensions, integral becomes

$$\int \frac{d^d l_E}{(2\pi)^d} \frac{l_E^2 + \ldots}{(l_E^2+M^2)^2} \sim \int^{\infty} dl_E \, l_E^{d-3} \rightarrow \text{converges for } d<2$$

Compute integrals as function of d → then take $d=4$ limit.
Divergences appear as terms $\sim \frac{1}{4-d}$

Here, compute $\Pi(q^2) - \Pi(0)$, which is finite.
After doing Feynman parameters & momentum shift:

$$i\Pi_{\mu\nu}(q) = -ie^2 \int_0^1 dx \int \frac{d^4 l_E}{(2\pi)^4} \frac{2 l_E^2 g_{\mu\nu} - 8x(1-x) q_\mu q_\nu + 4 g_{\mu\nu}(m^2 + x(1-x)q^2)}{(l_E^2+M^2)^2}$$

where $M^2 = m^2 - x(1-x)q^2$

$$= i(q^2 g_{\mu\nu} - q_\mu q_\nu)\Pi(q^2)$$

Pick out $q_\mu q_\nu$ term:

$$\Pi(q^2) - \Pi(0) = +e^2 \int_0^1 dx \int \frac{d^4 l_E}{(2\pi)^4} \left[\frac{8x(1-x)}{(l_E^2 + m^2 - x(1-x)q^2)^2} - \frac{8x(1-x)}{(l_E^2+m^2)^2} \right]$$

$$= \frac{e^2}{(2\pi)^4} 2\pi^2 \cdot 8 \int_0^1 dx \, x(1-x) \cdot \frac{1}{2} \log\left(\frac{m^2}{m^2 - x(1-x)q^2}\right)$$

consider $q^2 \gg m^2$ limit
$Q^2 =$

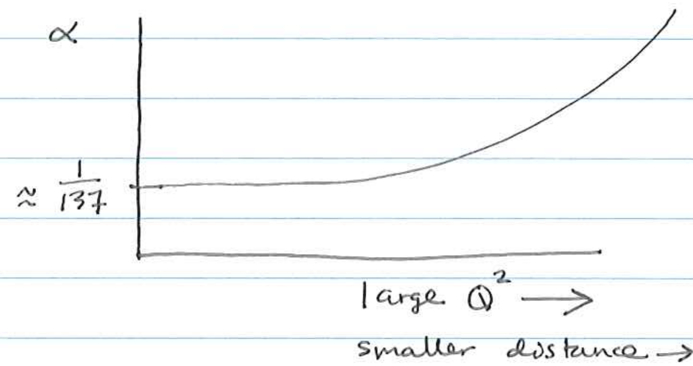
$$\cong \frac{2\alpha}{\pi} \int_0^1 dx \, x(1-x) \log\left(\frac{m^2}{-q^2 x(1-x)}\right)$$

$$\approx \frac{2\alpha}{\pi} \int_0^1 dx \, x(1-x) \left[-\log \frac{Q^2}{m^2} - \log x(1-x) \right]$$

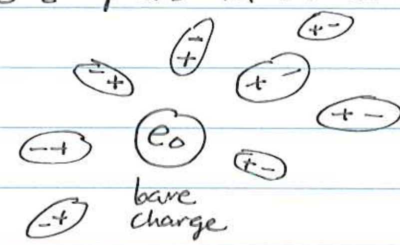
$$\approx -\frac{\alpha}{3\pi} \left(\log \frac{Q^2}{m^2} - \frac{5}{3} \right)$$

$$\Rightarrow \quad \alpha(Q^2) = \frac{\alpha(0)}{1 - \frac{\alpha(0)}{3\pi}\log\left(Q^2/Am^2\right)} \qquad A = e^{5/3}$$

Running coupling: electric charge depends on energy scale



Physical interpretation: virtual e^+e^- pairs in vacuum.



Bare charge is screened by virtual e^+e^- pairs.
At large distance, effective charge is $e(0)$ but at smaller & smaller distances sensitive to more of the unscreened bare (infinite) charge.

β-function: describes how α runs with energy (Q)

$$\beta(\alpha) = \frac{d\alpha}{d\log Q} = +\frac{2\alpha^2}{\pi} \qquad \text{at 1-loop order}$$

Also can be written as $\beta(e) = \dfrac{de}{d\log Q} = +\dfrac{e^3}{12\pi^2}$

+ Sign means α increases with energy.