Mesons and GUTs

22 January 2016

Due on Friday 5 February 2016

1 Meson Mixing

The CKM matrix in the Wolfenstein parametrization is (to order λ^4)

$$V_{CKM} = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix} = \begin{pmatrix} 1-\frac{1}{2}\lambda^2 & \lambda & A\lambda^3(\rho - i\eta) \\ -\lambda & 1-\frac{1}{2}\lambda^2 & A\lambda^2 \\ A\lambda^3(1-\rho-i\eta) & -A\lambda^2 & 1 \end{pmatrix} \tag{1}$$

where $\lambda \approx 0.2$ and A and $\rho^2+\eta^2$ are of order 1 experimentally. The advantage of this parametrization is that it summarizes the small sizes and hierachy of the off diagonal elements of V_{CKM}.

a) Prove the following identity using only the fact that V_{CKM} is unitary.

$$V_{us}V_{ud}^* + V_{cs}V_{cd}^* + V_{ts}V_{td}^* = 0 \tag{2}$$

This is often known as the “unitary triangle”: We can represent the addition of three complex numbers by three vectors on the complex plane. If the three vectors add up to zero then they form a triangle.

In fact, we can prove the generalized identity,

$$\sum_i V_{im}V_{in}^* = \delta_{mn} \tag{3}$$

where $i = u, c, t,\ m, n = d, s, b$.

b) Draw the **two** Feynman box diagrams responsible for kaon mixing ($K^0 \to \bar{K}^0$).

c) Write down the amplitude of one of the Feynman diagrams without evaluating it. We can parametrize the result as

$$i\mathcal{M} = \frac{G_F^2}{16\pi^2} \sum_{i,j=u,c,t} V_{id}^* V_{is} V_{jd}^* V_{js} f(m_i^2, m_j^2, m_s^2, m_d^2, m_w^2) \tag{4}$$

where f is some function of the masses.

d) Argue that if all quarks have the same mass, the above diagrams evaluate to zero. Glashow, Iliopoulos, and Maiani explained the suppression of kaon oscillations using the unitarity of the CKM matrix and predicted the fourth quark, the charm.

e) Investigate the loop integral, and argue that the result should take the form

$$i\mathcal{M} = \frac{G_F^2}{16\pi^2} \sum_{i,j=u,c,t} V_{id}^* V_{is} V_{jd}^* V_{js} m_i m_j h(x_i, x_j) \tag{5}$$

where $x_i = \frac{m_i^2}{m_W^2}$.

f) Argue that the dominant contribution to the matrix element comes from the graph that contains two virtual charm quarks. (Use the Wolfenstein parametrization of the CKM matrix and look up the masses of the quarks on the PDG.) [1]

g) The mass difference Δm between the mass eigenstates of the neutral K meson system is proportional to this amplitude:

$$\Delta m_K \approx \frac{G_F^2}{3\pi^2} f_K^2 m_K |V_{id}^* V_{is} V_{jd}^* V_{js}| m_i m_j \tag{6}$$

where $f_K = 100$ MeV is a constant that is related to the low energy effective theory of mesons. Estimate Δm_k and compare with the experiment data.

h) Draw the Feynman (box) diagrams responsible for $D^0 = c\bar{u}$, $B_d^0 = d\bar{b}$, $B_s^0 = s\bar{b}$ mixing.

i) For each of the mass differences, determine the virtual quark that provides the dominant contribution?

j) Estimate the mass difference for Δm_D and Δm_B. You may use $f_K \sim f_D \sim f_B$.

2 An $SU(5)$ Grand Unified Theory

In 1973, Georgi and Glashow proposed that the gauge group of the Standard Model, $SU(3)\times SU(2)\times U(1)$, was obtained by the spontaneous breaking of the simple Lie group $SU(5)$. This is one of a number of *grand unified theories* (GUT's) that seek to explain the quantum numbers of the quarks and leptons. $SU(5)$ is the group of 5×5 unitary matrices with determinant 1. The generators of this group are 5×5 matrices such that the transformations near the identity have the form

$$U \approx 1 + i\alpha^a t^a.$$

where the t^a are Hermitian and traceless matrices. There are 24 such matrices. We shall follow the convention of normalizing them to

$$\mathrm{tr}[t^a t^b] = \frac{1}{2}\delta^{ab}.$$

[1] Note that for CP violation, it is necessary to include the top quark contribution because V_{td} contains the complex phase.

a) Show that the matrix

$$T = \begin{pmatrix} -2 & & & & \\ & -2 & & & \\ & & -2 & & \\ & & & 3 & \\ & & & & 3 \end{pmatrix},$$

commutes with a subset of the 24 generators of $SU(5)$ that corresponds to the generators of $SU(3) \times SU(2) \times U(1)$. Find a matrix t^1 proportional to T that is a correctly normalized generator of $SU(5)$.

b) Consider an $SU(5)$ Yang-Mills theory with a Higgs field Φ in the 24 representation of $SU(5)$ (i.e., the adjoint representation where $\Phi = \Phi^a t^a$). In this representation, the covariant derivative is

$$D_\mu \Phi^a = \partial_\mu \Phi^a + g f^{abc} A^b_\mu \Phi^c \ \text{ or } \ D_\mu \Phi^a t^a = \partial_\mu \Phi^a t^a - ig \left[A^b_\mu t^b, \Phi^c t^c \right].$$

Give Φ the vacuum expectation value

$$\langle \Phi^a t^a \rangle = vT \ \text{ or } \ \langle \Phi^a \rangle = 2v\mathrm{tr}[Tt^a].$$

Show that 12 Yang-Mills bosons, corresponding to the generators of $SU(3) \times SU(2) \times U(1)$, stay massless, and the rest obtain equal masses.

c) Work out the quantum numbers of the five elements of the fundamental representation of $SU(5)$ under $SU(3) \times SU(2) \times U(1)$. Show that the five elements form an $SU(3)$ triplet that is an $SU(2)$ singlet and an $SU(2)$ doublet that is an $SU(3)$ singlet. Find the U(1) charge for each piece.

d) The representation matrices acting on the complex conjugate of this representation, the $\overline{5}$, are given by $t^a_{\overline{5}} = -(t^a_5)^T$. Why? Show that, if we introduce a left-handed fermion field in the $\overline{5}$, the five chiral fermion states have precisely the quantum numbers of the d_R and the L_L, with the recognition that we may need to rescale t^1 to obtain the conventional normalization for the hypercharges or conjugate one or both of the fields. Write, then, $Y = ct^1$. What is the value of c?

e) Another simple representation of $SU(5)$ is the 10-dimensional representation, given by anti-symmetric matrices with indices in the 5. The transformation of a 10 is

$$A_{ij} \to A_{ij} + i\alpha^a (t^a_{ik} A_{kj} + t^a_{jk} A_{ik}) \tag{7}$$

Show that the ten states of the 10 have precisely the quantum numbers of the chiral Standard Model fields Q_L, u_R, e_R (again possibly conjugated). Thus, an $SU(5)$ gauge theory with left-handed fermions in the $\overline{5}$ and 10 representations of $SU(5)$ unifies the Standard Model interactions and couplings.

f) Write the $SU(5)$ covariant derivative acting on the fundamental representation including the massless bosons only. If g_5 is the $SU(5)$ gauge coupling, the $SU(3)$, $SU(2)$, and $U(1)$ gauge couplings are given by $g_3 = g_2 = g_5$, $g_1 = g_5/c$. Show that these equations predict $\sin^2 \theta_W = \frac{3}{8}$.

These predictions for the gauge couplings and Weinberg angle are wrong experimentally. However, we know that coupling constants run. If the unification scale is sufficiently high (about 10^{15} GeV), different β-functions for the different gauge couplings can (almost) account for the observed differences at low energies. Despite its simplicity and usefulness in building more realistic (and complex) GUTs, the Georgi-Glashow model has a serious problem: it predicts that the proton decays much more quickly than is allowed by experiment.

g) *Optional*: Draw the parton level Feynman diagrams that give rise to proton decay in the Georgi-Glashow model. Use dimensional analysis to estimate the proton lifetime, assuming that the gauge bosons that mediate proton decay have a mass of 10^{15} GeV. Compare your estimate with the PDG bounds on the proton lifetime.