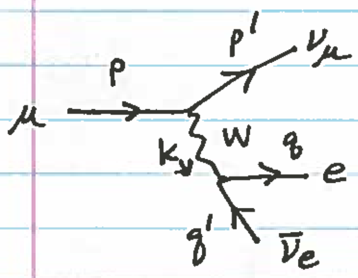
Effective field theory for the weak interaction

For weak processes with energy $E \ll M_W, M_Z$, we don't need to use the full SM Lagrangian with W, Z as degrees of freedom. We can use an effective Lagrangian where we "integrate out" the W, Z, expressing their interactions as a nonrenormalizable operator.

example: μ-decay

$$i\mathcal{M}(\mu \to \nu_\mu e \bar{\nu}_e) = \bar{u}_e \left(\frac{-ig}{\sqrt{2}}\right)\gamma^\mu P_L v_{\nu_e} \frac{-i}{k^2 - m_W^2}\left(\eta_{\mu\nu} - \frac{k_\mu k_\nu}{m_W^2}\right)$$

$$\times\ \bar{u}_{\nu_\mu}\left(\frac{-ig}{\sqrt{2}}\right)\gamma^\nu P_L u_\mu$$



Note: using the Dirac equation (consider $k_\mu k_\nu$ term)

$$\bar{u}_e \not{k} P_L v_{\nu_e} = \bar{u}_e (\not{q} + \not{q}') P_L v_{\nu_e} = m_e \bar{u}_e P_L v_{\nu_e}$$

$$\bar{u}_{\nu_\mu} \not{k} P_L u_\mu = m_\mu \bar{u}_{\nu_\mu} P_R u_\mu$$

$$\Rightarrow k_\mu k_\nu \text{ terms are suppressed by } \left(\frac{m_e m_\mu}{m_W^2}\right) \sim 10^{-8}$$

Also $k^2 \ll m_W^2$

$$i\mathcal{M} = \frac{i g^2}{8 m_W^2} \bar{u}_e \gamma^\mu (1-\gamma^5) v_{\nu_e}\ \bar{u}_{\nu_\mu} \gamma_\mu (1-\gamma^5) u_\mu$$

This can be expressed as an operator

$$\mathcal{L}_{eff} = \frac{G_F}{\sqrt{2}} \bar{e} \gamma^\mu (1-\gamma^5)\nu_e \cdot \bar{\nu}_\mu \gamma_\mu (1-\gamma^5)\mu + h.c.$$

where $\frac{G_F}{\sqrt{2}} = \frac{g^2}{8m_W^2}$ is "Fermi's constant". mass dim -2.

Charge current interactions can be ~~...~~ described by a four-fermion effective theory (Fermi theory)

$$\mathcal{L}_{eff}^{cc} = \frac{G_F}{\sqrt{2}} \sum_{i,j} \bar{e}^i \gamma^\mu (1-\gamma^5) \nu^i \, \bar{\nu}^j \gamma_\mu (1-\gamma^5) e^j + h.c.$$
(leptonic)

$$+ \frac{G_F}{\sqrt{2}} \sum_{i,j,k} V_{ij} \, \bar{u}^i \gamma^\mu (1-\gamma^5) d^j \, \bar{e}^k \gamma_\mu (1-\gamma^5) \nu^k + h.c.$$
(semileptonic)

$$+ \frac{G_F}{\sqrt{2}} \sum_{ijkl} V_{ij} V_{kl}^* \, \bar{u}^i \gamma^\mu (1-\gamma^5) d^j \, \bar{d}^l \gamma_\mu (1-\gamma^5) u^k + h.c.$$
(hadronic)

$SU(2)_L \times U(1)_Y$ gauge theory explains why all types of interactions have the same coupling $G_F/\sqrt{2}$ (modulo CKM) and why all interaction have universal Dirac structure $\gamma^\mu(1-\gamma^5) \otimes \gamma_\mu(1-\gamma^5)$ (known as $(V-A)\times(V-A)$).

Also can write down $\mathcal{L}_{eff}^{N.C}$ by integrating out Z boson.

Note: $\frac{G_F}{\sqrt{2}} = \frac{g^2}{8m_W^2} = \frac{1}{2v^2}$ using $m_W = \frac{gv}{2}$

$\Rightarrow$ Higgs vev fixed by measured value for G_F (from e.g. μ dec
$G_F \simeq 1.166 \times 10^{-5}\ \text{GeV}^{-2}$

$$\boxed{v = (\sqrt{2}\, G_F)^{-1/2} \simeq 246\ \text{GeV}}$$

Beware there is an alternate convention $H = \begin{pmatrix} 0 \\ v + \frac{h}{\sqrt{2}} \end{pmatrix}$
In this case $v \simeq 174$ GeV. Once v is fixed in terms of G_F, the convention doesn't matter.

<u>Accidental symmetries</u> : a global symmetry that results because it is not possible to have any terms in the (renormalizable) Lagrangian that violate the symmetry (due to restriction of gauge invariance)

In the SM, there are four accidental $U(1)$ symmetries: baryon number (B) and lepton flavor (L_e, L_μ, L_τ).

- B: all quark fields in $\mathcal{L}$ have the structure

$$\bar{q}^i \Gamma q^j \quad \text{where } \Gamma = \text{combination of Dirac/SU(3) color matrix.}$$

$\mathcal{L}$ is invariant under

$$q^i \to e^{i\theta} q^i$$
$$\bar{q}^i \to \bar{q}^i e^{-i\theta}$$

while other SM fields are invariant. This is a $U(1)_B$ transformation. q^i has $B = 1/3$, while all other fields have $B=0$. Lightest baryon (proton) is stable.

- L_e: $U(1)_{L_e}$ transformation $\left.\begin{array}{l} e \to e^{i\theta} e \\ \nu_e \to e^{i\theta} \nu_e \end{array}\right\} L_e = 1$

  other fields invariant.

- L_μ: $U(1)_{L_\mu}$ transform: $\left.\begin{array}{l} \mu \to e^{i\theta} \mu \\ \nu_\mu \to e^{i\theta} \nu_\mu \end{array}\right\} L_\mu = 1$

  other fields invariant

- L_τ: similar for τ, ν_τ fields. $\} L_\tau = 1$

B, L_e, L_μ, L_τ are all conserved in the SM (with massless neutrinos). It is also useful to consider total lepton number $L = L_e + L_\mu + L_\tau$.

$U(1)_L$ is also a symmetry: rotate all lepton fields $\psi \to e^{i\theta}\psi$ for $\psi = e, \mu, \tau, \nu_i$

(Lepton flavor [and maybe L] symmetries are violated by neutrino masses?

Parameters of the SM

3 Gauge couplings: g, g', g_s

2 Higgs parameters: mass $m_h^2 = 2\mu^2$, quartic λ } $m_{W,Z}$, v fixed.

9 Fermion masses: $m_e, m_\mu, m_\tau, m_u, m_c, m_t, m_d, m_s, m_b$

4 CKM matrix parameters: V_{ij} (how many?)

(Also additional parameter θ_{QCD}, as far as we can tell $\theta_{QCD} \approx 0$

Everything* can be computed in terms of these parameters.

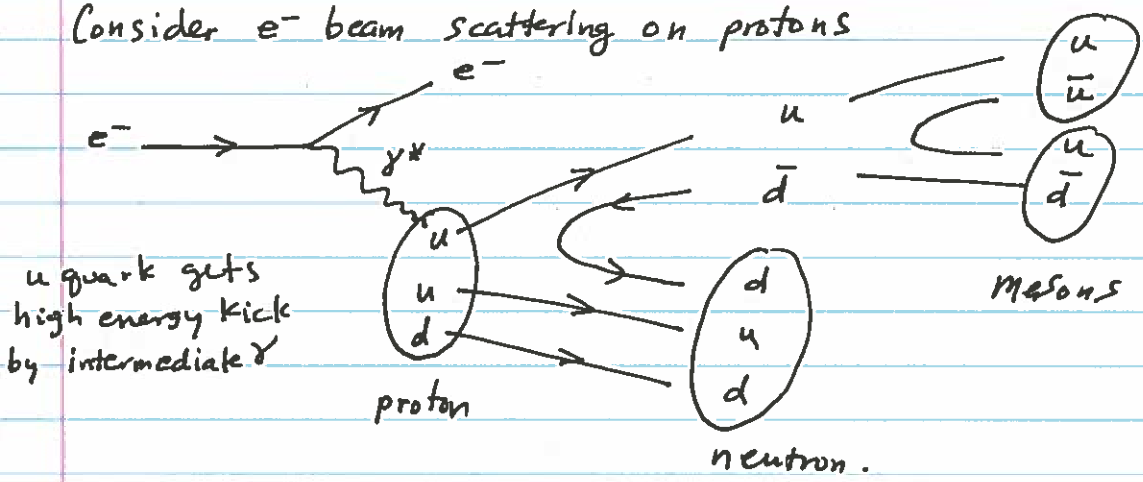
*except low energy QCD & neutrino masses...

Deep Inelastic Scattering (DIS)

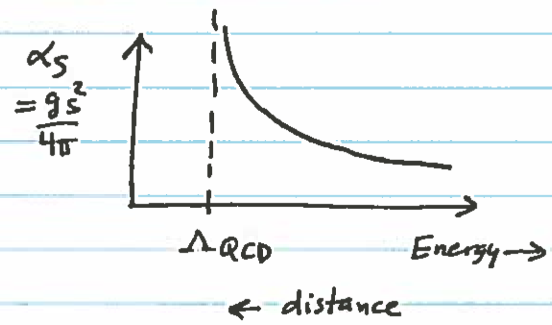
Hadrons are made from quarks — how do we know this?

We can't unbind a free quark from a hadron. Putting in energy to separate a quark just produces more hadrons because it is energetically favorable to produce quark-antiquark pairs than have a free quark. (confinement)

Consider e^- beam scattering on protons



This process is called *fragmentation* or *hadronization*.
It results from the fact that the β-function for QCD is negative, with a Landau pole at $\Lambda_{QCD} \sim 200$ MeV.



(Ch.16 Peskin & Schroeder)

Hadronization is a nonperturbative effect that occurs when $\frac{\alpha_s}{4\pi} \sim 1$. The distance scale for hadronization is $\sim \Lambda_{QCD}^{-1} \sim$ fm $= 10^{-15}$

(Note: this is also the proton radius)

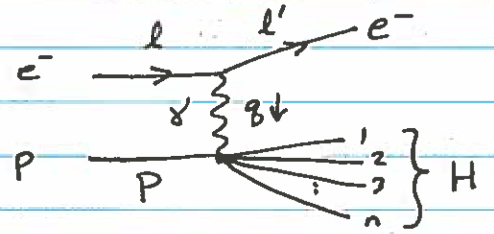
The flipside of confinement is asymptotic freedom.
At distances $\ll \Lambda_{QCD}^{-1} \sim$ fm, quarks behave as free fermions & perturbation theory is valid ($\alpha_s \ll 1$).

These facts imply an important result known as factorization:

What happens at small distances ($\ll$ fm) isn't affected by dynamics over large distances ($\sim$ fm).

Factorization is the reason it is possible to make precision calculations for high energy processes involving quarks even though the initial and/or final states are hadrons. (e.g. LHC)

Before we see how this works, let's consider $e^- p$ scattering in general



H = final hadronic state.
elastic scattering: H = p (proton is not "broken up")
inelastic scattering: H = multiple hadronic states labeled $1, \dots, n$. (proton "broken up")

4-momenta: ℓ^μ = initial e^-
ℓ'^μ = outgoing e^-
P^μ = initial proton
$P'^\mu = \sum_{i=1}^{n} P_i'^\mu$ = total 4-momentum of all hadronic final states.

Also, we'll set $m_e \to 0$. And $q^\mu = \ell^\mu - \ell'^\mu$ = photon 4-momentum.

Helpful to define some kinematic variables:

(1) $Q^2 = -q^2$ = momentum transfer (squared)
Note: $Q^2 > 0$. $Q^2 = -(\ell-\ell')^2 = 2\ell\cdot\ell' = 2E_\ell E_{\ell'}(1-\cos\theta)$

where θ is outgoing e^- scattering angle.

(2) $\nu = \frac{P\cdot q}{m_p}$. In rest frame of initial proton, $P = (m_p, 0, 0$

$$\Rightarrow \nu = \frac{m_p(E_\ell - E_{\ell'})}{m_p} = E_\ell - E_{\ell'}$$

ν is the energy transfer in the p rest frame.

(3) $y = \frac{P\cdot q}{P\cdot \ell}$ In p rest frame, $P\cdot\ell = m_p E_\ell$

$\Rightarrow y = \frac{\nu}{E_\ell} = 1 - E_{\ell'}/E_\ell$ is the fractional energy transfer in the p rest frame.

(4) $x = \frac{-q^2}{2P\cdot q} = \frac{Q^2}{2P\cdot q} = \frac{Q^2}{2m_p\nu}$ "Bjorken x"

Note: all the variables are made from Lorentz scalars $\rightarrow$ same in any frame

To see the meaning of x, let's consider elastic scattering (H = proton). Also note momentum conservation $P + \ell = P' + \ell'$.

$$P'^2 = m_p^2 = (P + \ell - \ell')^2 = (P+q)^2 = m_p^2 + 2P\cdot q + q^2$$

$$\Rightarrow \quad 2P\cdot q = -q^2 = Q^2 \quad \Rightarrow \quad \frac{Q^2}{2P\cdot q} = 1 = x$$

Elastic scattering $\rightarrow$ $x = 1$.

Now consider inelastic scattering:

Note: $P'^2 = (P_1' + P_2' + \dots + P_n')^2 = \sum_{i=1}^{n} m_i^2 + \text{cross terms} \sim 2P_1'\cdot P_2'$, etc.
$> m_p^2$

So $P'^2 - m_p^2 = 2P\cdot q + q^2 = 2P\cdot q\,(1-x) > 0$

$P\cdot q > 0$ (since $\nu > 0$), then we have $x < 1$.

Also $Q^2 > 0 \Rightarrow$ allowed range for x: $0 < x < 1$.

Recall:
For $2\to 2$ scattering, the differential cross section $\frac{d\sigma}{d\cos\theta}$ has just one independent kinematic variable, e.g. $\cos\theta$ or t. (in addition to the total CM energy^2 $= s$).

Here, elastic scattering ($x=1$) has one kinematic variable which can be either Q^2, ν, y (or $\cos\theta$) + $s=(P+\ell)^2$.

But inelastic scattering ($0<x<1$) has two independent kinematic variables: x and either Q^2, ν, or y (in addition to $s=(P+\ell)^2$)

Now let's compute the cross section:

Inclusive scattering : $\sigma(e^-p\to e^-X) = \sum_H \sigma(e^-p\to e^-H)$
Sum over all hadronic final states H. ($X=$ any hadronic final state)
Work in deep inelastic regime: $\sqrt{s} \gg m_p$, $\nu \gg m_p$.

$$\sigma(e^-p\to e^-X) = \sum_H \sigma(e^-p\to e^-H)$$

$$= \sum_H \frac{1}{2E_\ell\, 2E_P\, |\underline{v}_{rel}|} \int \frac{d^3\ell'}{(2\pi)^3}\frac{1}{2E_\ell'} \prod_{i=1}^{n}\int \frac{d^3P_i'}{(2\pi)^3}\frac{1}{2E_{P_i'}}$$

$$\times (2\pi)^4\,\delta^4(\ell+P-\ell'-\sum_i P_i') \times \frac{1}{4}\sum_{\text{spins}} |\mathcal{M}(e^-p\to e^-H)|^2$$

Matrix element:

$$i\mathcal{M}(e^-p\to e^-H) = \bar{u}(\ell')\, ie\gamma^\mu u(\ell)\; \frac{-i}{q^2}\langle H|\, ie\, J_\mu^{em}\,|P\rangle$$

where $e\,j_\mu^{em} = e\sum_q Q_q\,\bar{q}\gamma^\mu q$ is the EM current that couples to γ field A_μ in $\mathcal{L}$.

But we don't know how j_μ^{em} connects $\langle H|$ and $|P\rangle$, depends on dynamics of hadronization.

$$\frac{1}{4}\sum_{\text{spins}} |\mathcal{M}(e^-p\to e^-H)|^2 = \frac{e^4}{q^4}\frac{1}{4}\,\mathrm{Tr}[\not{l}'\gamma^\mu \not{l}\gamma^\nu]\langle H|j^{em}_\mu|p\rangle\langle p|j^{em}_\nu|H\rangle$$

$$\sigma(e^-p\to e^-X) = \frac{1}{2E_l\,|v_{rel}|}(2\pi)\int\frac{d^3l'}{(2\pi)^3}\frac{1}{2E_{l'}}\,\frac{e^4}{q^4}\,\frac{1}{2}\mathrm{Tr}[\not{l}'\gamma^\mu\not{l}\gamma^\nu]$$

$$\left.\begin{aligned}&\times\ \frac{1}{2E_p}\sum_H\prod_{i=1}^{n}\int\frac{d^3p_i'}{(2\pi)^3}\frac{1}{2E_{p_i'}}\,(2\pi)^3\delta^4(l+P-l'-\textstyle\sum_i P_i')\\ &\times\ \frac{1}{2}\sum_{\text{spins}}\langle H|j^{em}_\mu|p\rangle\langle p|j^{em}_\nu|H\rangle\end{aligned}\right\}W_{\mu\nu}$$

Represent all the unknown hadronic physics by a tensor $W_{\mu\nu}$. Also define leptonic tensor

$$L_{\mu\nu} = \frac{1}{2}\mathrm{Tr}[\not{l}'\gamma_\mu\not{l}\gamma_\nu] = 2(l_\mu l'_\nu + l_\nu l'_\mu - l\cdot l'\,\eta_{\mu\nu})$$

Use Lorentz invariance to restrict the form of $W_{\mu\nu}$. Can only depend on 4-momenta P_μ, q_μ.

It must be a linear combination of all possible ways to have a two-index object.

$$W_{\mu\nu} \supset \underbrace{\eta_{\mu\nu},\ P_\mu P_\nu}_{\text{OK.}},\ \underbrace{q_\mu q_\nu,\ P_\mu q_\nu,\ q_\mu P_\nu}_{\text{will vanish when contracted with } L^{\mu\nu}},\ \underbrace{\varepsilon_{\mu\nu\alpha\beta}P^\alpha q^\beta}_{\text{Parity violating, not allowed for EM current}}$$

So most general form for $W_{\mu\nu}$ is:

$$W_{\mu\nu}(p,q) = -W_1\,\eta_{\mu\nu} + W_2\,\frac{P_\mu P_\nu}{m_p^2}$$

$W_{1,2}$ are Lorentz scalars. They depend on the Lorentz scalar quantities we have defined: e.g. $W_{1,2}(Q^2, x)$.

Then we have: $\sigma(e^- p \to e^- X) = \frac{1}{2E_\ell |v_{rel}|} (2\pi) \int \frac{d^3\ell'}{(2\pi)^3 2E_{\ell'}} \frac{e^4}{q^4} L^{\mu\nu} W_{\mu\nu}$

Work in "lab frame" where initial proton is at rest. $|\underline{v}_{rel}| = 1$.
Also assume $E_\ell, E_{\ell'}, \nu \gg m_p$. CM energy squared is
$$s = (P+\ell)^2 = m_p^2 + 2P\cdot\ell = m_p^2 + 2m_p E_\ell \approx 2m_p E_\ell .$$

$$L^{\mu\nu} W_{\mu\nu} = 2(\ell^\mu \ell'^\nu + \ell^\nu \ell'^\mu - \eta^{\mu\nu}\,\ell\cdot\ell')\left(-W_1 \eta_{\mu\nu} + \frac{P_\mu P_\nu}{m_p^2} W_2\right)$$

$$= 4\ell\cdot\ell' W_1 + W_2 \left[\frac{4(P\cdot\ell)(P\cdot\ell')}{m_p^2} - 2\ell\cdot\ell'\right]$$

$$= 2Q^2 W_1 + W_2\left[4E_\ell E_{\ell'} - Q^2\right] = 2E_\ell E_{\ell'}\Big[\underbrace{2W_1(1-\cos\theta)}_{\text{large angle}} + \underbrace{W_2(1+\cos\theta)}_{\text{small angle}}$$

$$= 4m_p\nu x W_1 + (4E_\ell E_{\ell'} - 2m_p\nu x) W_2$$

2nd term << 1st term

$$= 4m_p E_\ell\, y\, x W_1 + 4E_\ell^2 (1-y) W_2 \qquad 1-y = E_{\ell'}/E_\ell$$

$$\sigma(e^- p \to e^- X) = \frac{1}{2E_\ell}\frac{1}{4\pi^2}\int \frac{E_{\ell'}^2 dE_{\ell'}\, d\cos\theta}{2E_{\ell'}} (2\pi)\frac{e^4}{q^4} L^{\mu\nu} W_{\mu\nu}$$

↘ convert to integral over $\int dx\,dy$

$$dy = -\frac{dE_{\ell'}}{E_\ell} \Rightarrow dE_{\ell'} = -dy\, E_\ell$$

$$dx = \frac{\partial}{\partial\cos\theta}\left(\frac{2E_\ell E_{\ell'}(1-\cos\theta)}{2m_p\nu}\right) d\cos\theta = \frac{E_\ell E_{\ell'}}{m_p\nu}\, d\cos\theta$$

$$\Rightarrow d\cos\theta = \frac{m_p\nu}{E_\ell E_{\ell'}}\, dx$$

$$E_{\ell'}\, dE_{\ell'}\, d\cos\theta = E_{\ell'}\frac{dy\, E_\ell\; m_p\nu}{E_\ell E_{\ell'}}\, dx = m_p\nu\, dx\, dy$$

$$\sigma(e^- p \to e^- X) = \frac{1}{8\pi E_\ell}\int dx\,dy\, \frac{e^4}{Q^4}\left[4m_p^2 E_\ell\, y x \nu W_1 + 4E_\ell^2(1-y) W_2 m_p \nu\right.$$

$$\cancel{= 2\pi\alpha^2 \int dx\,dy\,\frac{1}{Q^4}\left[2sy^2 x (W_1\nu) + \right.}$$

$$= \int dx\,dy\, \frac{2\pi\alpha^2}{Q^4}\Big[s x y^2 \underbrace{(2m_p W_1)}_{F_1^e} + 2s(1-y)\underbrace{(\nu W_2)}_{F_2^e}\Big]$$

It is customary to define two new functions:

$$F_1^e(x,Q^2) = 2m_p W_1(x,Q^2)$$
$$F_2^e(x,Q^2) = \nu W_2(x,Q^2)$$

Final result

$$\sigma(e^- p \to e^- X) = \int dx\,dy \frac{2\pi\alpha^2 s}{Q^4}\left[F_1^e(x,Q^2)\,xy^2 + 2F_2^e(x,Q^2)(1-y)\right]$$

or can be expressed as diff. cross section:

$$\frac{d^2\sigma(e^- p \to e^- X)}{dx\,dy} = \frac{2\pi\alpha^2 s}{Q^4}\left[F_1^e(x,Q^2)\,xy^2 + 2F_2^e(x,Q^2)(1-y)\right]$$

So inelastic scattering depends on two unknown functions $F_{1,2}^e(x,Q^2)$.

Now let's return to the idea of factorization & computing the same cross section from scattering off of quarks.

σ factorizes as a product:

$$\sigma(e^- p \to e^- H) = \sum_{q,p} \underbrace{\text{Prob}(q,p)}_{\substack{\text{probability of finding}\\ \text{quark } q \text{ with momentum}\\ p \text{ inside proton.}\\ \text{(large dist.)}}} \times \underbrace{\hat{\sigma}(e^- q \to e^- q)}_{\substack{\text{cross section}\\ \text{for scattering}\\ \text{off of quark } q\\ \text{with mom. } p.\\ \text{(short dist.)}}} \times \underbrace{\text{Prob}(H)}_{\substack{\text{prob. for final}\\ \text{state to hadronize}\\ \text{into } H.\\ \text{(large dist.)}}}$$

Then sum (integrate) over all possible q and p.

We don't know Prob(H), but if we consider an inclusive final state we don't need to.

$$\sigma(e^- p \to e^- X) = \sum_H \sigma(e^- p \to e^- H) = \sum_{q,p} \text{Prob}(q,p)\,\hat{\sigma}(e^- q \to e^- q) \sum_H \text{Prob}(H)$$

We know that $\sum_H \text{Prob}(H) = 1$ by confinement.

So we have: $\sigma(e^- p \to e^- X) = \sum \text{Prob}(q, p)\ \hat{\sigma}(e^- q \to e^- q)$

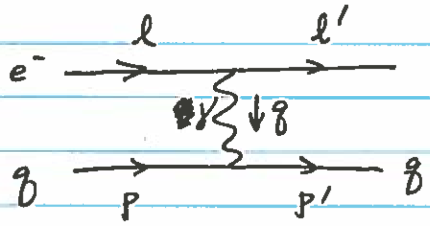
This setup is known as the <u>parton model</u>. It predates QCD, but is consistent with ideas of factorization & asymptotic freedom from QC

<u>Partons</u> are the constituents inside hadrons, i.e. quarks, antiquarks and gluons. Since we are considering scattering via photon, only need to consider quark/antiquark partons since gluons have no EM charge.

Parton-level quantities are denoted by hats. (e.g. $\hat{\sigma}$)

Except for the initial probability of finding q (with a given momentum) inside the proton, $\sigma(e^- p \to e^- X)$ only depends on short distance physics which can be computed using perturbation theory.

Parton-level cross section $\hat{\sigma}$: assume quarks are massless.



$$i\mathcal{M} = \bar{u}(\ell')(ie\gamma^\mu)u(\ell)\ \frac{-i}{q^2}\ \bar{u}(p')(-ieQ_q\gamma_\mu)u(p)$$

$$\frac{1}{4}\sum|\mathcal{M}|^2 = \frac{1}{4}\frac{e^4}{q^4}Q_q^2\ \text{Tr}[\not{\ell}'\gamma^\mu\not{\ell}\gamma^\nu]\,\text{Tr}[\not{p}'\gamma_\mu\not{p}\gamma_\nu]$$

$$= \frac{4e^4}{q^4}Q_q^2\,(\ell\cdot p\ \ell'\cdot p' + \ell\cdot p'\ \ell'\cdot p)$$

Mandelstam variables:

$$\hat{s} = (p+\ell)^2 = (p'+\ell')^2 = 2p\cdot\ell = 2p'\cdot\ell'$$
$$\hat{t} = (\ell-\ell')^2 = q^2$$
$$\hat{u} = (\ell-p')^2 = -2\ell\cdot p' = (\ell'-p)^2 = -2\ell'\cdot p$$

also note $\hat{s}+\hat{t}+\hat{u}=0$

$$\frac{1}{4}\sum_{spins} |\mathcal{M}|^2 = \frac{2e^4 Q_q^2}{\hat{t}^2}\left(\hat{s}^2 + \hat{u}^2\right)$$

Compute cross section in CM frame:

$$\hat{\sigma}(eq\to eq) = \frac{1}{2E_\ell \cdot 2E_p |v_{rel}|} \int \frac{d^3\ell'}{(2\pi)^3}\frac{1}{2E_{\ell'}} \int \frac{d^3p'}{(2\pi)^3}\frac{1}{2E_{p'}} (2\pi)^4 \delta^4(\ell+p-\ell'-p')$$

$$\times \frac{1}{4}\sum|\mathcal{M}|^2$$

$$= \frac{1}{2\hat{s}^2}\int \frac{d^3\ell'}{(2\pi)^3}(2\pi)\,\delta(\sqrt{\hat{s}} - 2E_{\ell'}) \cdot \frac{1}{4}\sum|\mathcal{M}|^2$$

$$= \frac{1}{8\pi^2\hat{s}^2}\int E_{\ell'}^2 dE_{\ell'} \int d\Omega \; \frac{1}{2}\delta\left(E_{\ell'} - \frac{\sqrt{\hat{s}}}{2}\right) \cdot \frac{1}{4}\sum|\mathcal{M}|^2$$

$$= \frac{1}{4\pi\hat{s}^2}\; \frac{1}{2}\left(\frac{\sqrt{\hat{s}}}{2}\right)^2 \int d\cos\theta \;\; \frac{1}{4}\sum|\mathcal{M}|^2$$

$$\frac{d\hat{\sigma}}{d\cos\theta} = \frac{1}{32\pi\hat{s}} \cdot \frac{2e^4Q_q^2}{\hat{t}^2}(\hat{s}^2+\hat{u}^2) \qquad e^4 = (\alpha\, 4\pi)^2 = \alpha^2 16\pi^2$$

$$= \frac{\pi\alpha^2 Q_q^2}{\hat{s}\hat{t}^2}\left(\hat{s}^2+\hat{u}^2\right)$$

Note: $\hat{t} = (\ell'-\ell)^2 = -2\ell\cdot\ell' = -2E_\ell^2 + 2|\ell||\ell'|\cos\theta$

$$= -\frac{1}{2}\hat{s}\,(1-\cos\theta)$$

$$d\hat{t} = \frac{1}{2}\hat{s}\; d\cos\theta$$

$$\frac{d\sigma}{d\hat{t}} = \frac{d\sigma}{d\cos\theta}\frac{d\cos\theta}{d\hat{t}} = \frac{d\sigma}{d\cos\theta}\cdot\frac{2}{\hat{s}}$$

$$\frac{d\hat{\sigma}}{d\hat{t}} = \frac{2\pi\alpha^2Q_q^2}{\hat{s}^2\hat{t}^2}\left(\hat{s}^2+\hat{u}^2\right)$$

Now let's go from parton-level to hadron-level:

We have (still) $q^2 = -Q^2 = \hat{t}$

p = 4-momentum of initial quark
P = 4-momentum of proton.
The quark that interacts with γ ~~[illegible]~~ (the struck quark) carries a fraction x of the total momentum.

So $p = x\,P$, where $0 < x < 1$.

We can solve for x: since $m_q = 0$, $p^2 = 0$, $p'^2 = 0$

$$p'^2 = 0 = (p + l - l')^2 = p \cdot q + q^2 = x\,P \cdot q + q^2$$

$$\Rightarrow \quad x = \frac{-q^2}{2P \cdot q} = \frac{Q^2}{2P \cdot q}$$

same x as above.
momentum fraction of struck quark

Lastly, we can relate $\hat{s}$ to s by:

$$\hat{s} = (l + p)^2 = 2\,l \cdot p = 2\,l \cdot P\,x = s\,x$$

And also: $y = \dfrac{P \cdot q}{P \cdot l} = \dfrac{p \cdot q}{p \cdot l} = \dfrac{x\,p \cdot p'}{x\,p \cdot l} = \dfrac{-\hat{t}}{\hat{s}} \Rightarrow \hat{t} = -y\,x\,s$

$$\hat{u} = -\hat{s} - \hat{t} = -s\,x\,(1 - y)$$

So can write differential cross section as:

$$\frac{d\hat{\sigma}}{dy} = \frac{d\hat{\sigma}}{d\hat{t}} \left| \frac{d\hat{t}}{dy} \right| = \frac{2\pi\alpha^2 Q_q^2}{Q^4}\left(y^2 + 2(1-y)\right)$$

Lastly, we need to include the probability for finding a quark q inside the initial proton. The probability to find q with momentum fraction between x and $x+dx$ is given by

$$f_q(x)\,dx$$

where $f_q(x)$ is the parton distribution function. (pdf).

Thus the total cross section is

$$\sigma(e^-p \to e^-X) = \underbrace{\sum_q \int_0^1 dx\, f_q(x)}_{\substack{\text{sums over all possible}\\ \text{quarks } q \text{ and momenta with}\\ \text{the appropriate probability}\\ \text{function (the pdf)}}}\, \hat{\sigma}(e^-q \to e^-q)$$

or we can write diff. cross section:

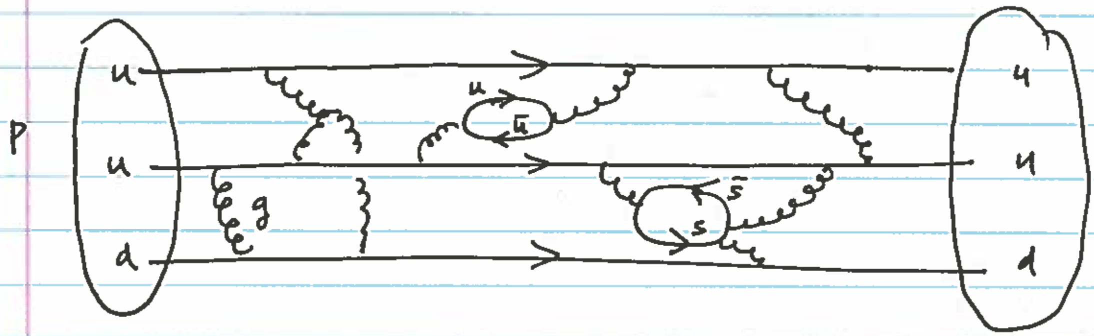
$$\frac{d\sigma(e^-p \to e^-X)}{dx} = \sum_q f_q(x)\, \hat{\sigma}(e^-q \to e^-q)$$

or doubly-diff cross section

$$\frac{d^2\sigma(e^-p \to e^-X)}{dx\,dy} = \sum_q f_q(x)\, \frac{d\hat{\sigma}}{dy} \quad \Big\}\ \text{exactly what we computed before.}$$

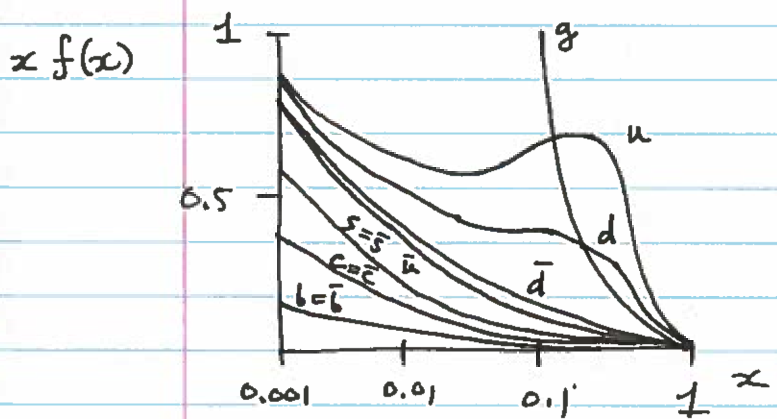
Parton distribution functions: (pdfs)

proton = uud. But this is too simplistic.
uud are the valence quarks. There are also $q\bar{q}$ pairs that fluctuate from the vacuum - these are sea quarks.



A high energy photon can scatter with either valence or sea quarks.
$\sum_q f_q(x)$ sums over all quark and antiquark pdfs.

The gluon also has a pdf $f_g(x)$, but we don't need to include it here since photons don't scatter off of gluons (neutral).



High momentum partons in p are mostly valence quarks, low momentum partons are mostly gluons.
Heavy quarks do exist inside p, but are rare.

Double-differential cross section:

$$\frac{d^2\sigma(e^-p\to e^-X)}{dx\,dy} = \sum_q f_q(x)\,\frac{d\hat\sigma}{dy} = \sum_q f_q(x)\,\frac{d\hat\sigma}{d\hat t}\left|\frac{d\hat t}{dy}\right|$$

$$\left|\frac{d\hat t}{dy}\right| = xs$$

$$\frac{d\hat\sigma}{d\hat t} = \frac{2\pi\alpha^2 Q_q^2}{x^2 s^2 Q^4}\left(x^2s^2 + s^2x^2(1-y)^2\right)$$

$$= \frac{2\pi\alpha^2 Q_q^2}{Q^4}\left(y^2 + 2(1-y)\right)$$

Final result is:

$$\frac{d^2\sigma(e^-p\to e^-X)}{dx\,dy} = \cancel{\frac{2\pi\alpha^2 s}{Q^4}\sum_q} \sum_q x\,f_q(x)\,\frac{2\pi\alpha^2 Q_q^2}{Q^4}\left(y^2+2(1-y)\right)$$

For future reference: define two functions $F_{1,2}^e(x)$ such that

$$\frac{d^2\sigma(e^-p\to e^-X)}{dx\,dy} = \frac{2\pi\alpha^2 s}{Q^4}\left[F_1^e(x)\,x\,y^2 + 2F_2^e(x)(1-y)\right]$$

So we have $F_1^e(x)\,x = F_2^e(x) = \sum_q x\,Q_q^2 f_q(x)$

$$= x\left(\tfrac{2}{3}\right)^2\left(f_u(x)+f_{\bar u}(x)\right)$$
$$+\, x\left(-\tfrac{1}{3}\right)^2\left(f_d(x)+f_{\bar d}(x)\right)$$
$$+\, x\left(-\tfrac{1}{3}\right)^2\left(f_s(x)+f_{\bar s}(x)\right) + \dots$$

~~Q^2 can also expressed in terms of x, y: $Q^2 = sxy$.~~
~~By measuring how $\frac{d^2\sigma}{dx\,dy}$ depends on y for fixed x, the two terms $F_{1,2}^e$ can disentangled be measured separately.~~

This is the important result: general calculation $\frac{d^2\sigma}{dx\,dy}$ depended on two functions $F_{1,2}^{e}(x,Q^2)$, which were functions of x, Q^2.

The prediction from parton model (proton made from spin ½ quarks) is that (1) $F_{1,2}^{e}$ only depend on x, not Q^2: <u>Bjorken scaling</u>

and (2) that $x F_1^e(x) = F_2^e(x)$ <u>Callan-Gross relation</u>.

If quarks were spin-0, then we would get $F_1^e(x) = 0$ instead.

DIS experiments we done at SLAC in late 1960's-early 1970's. Observation of Bjorken scaling was one of key pieces of evidence for existence of quarks. QCD is also consistent with Bjorken scaling (by asymptotic freedom & factorization), but only at lowest order in α_s (that is, α_s^0). Higher order terms in QCD at order α_s lead to "scaling violations" that can be computed and agree with data. ($F_{1,2}^{e}$ do depend on Q^2 in way that can be computed at $\mathcal{O}(\alpha_s)$.)