Hadron spectroscopy & group theory

Quarks are confined in color-neutral mesons $\sim (q_1 \bar{q}_2)$ or baryons $\sim (q_1 q_2 q_3)$. The typical mass scale for QCD is set by $\sim \Lambda_{QCD} \sim 200$ MeV.

Light quark hadrons: only containing (u, d, s) [or just (u,d)]
Heavy quark hadrons: one or more heavy quarks (c, b)
t-quark is irrelevant for hadron states since t decays very rapidly, before hadronization occurs (although virtual t can be very important in certain contexts).

Mesons: (masses in MeV, lifetimes in sec.)

	State	quark content	mass	lifetime	J^P	leading decay(s)
pions	π^0	$u\bar{u} - d\bar{d}$	135	8.5×10^{-17}	0^-	$\pi^0 \to \gamma\gamma$ (EM)
	$\pi^\pm$	$u\bar{d}$, $d\bar{u}$	140	2.6×10^{-8}	0^-	$\pi^+ \to \nu_\mu \mu^+$ (Weak)
kaons	$K^\pm$	$u\bar{s}$, $s\bar{u}$	494	1.2×10^{-8}	0^-	$\pi\pi$, $\mu^\pm \nu_\mu$ (W)
	$K^0, \bar{K}^0$	$d\bar{s}$, $s\bar{d}$	498	$\sim 10^{-8}$ or 10^{-10}	0^-	2π, 3π, (W) $\pi e \nu_e$, $\pi \mu \nu_\mu$
	f_0	$\pi\pi$ state?	~ 550	$\sim 10^{-27}$ s	0^+	$\pi\pi$
	η	$\sim u\bar{u} + d\bar{d} - 2s\bar{s}$	548	6.6×10^{-22} s	0^-	$\eta \to \gamma\gamma$ (EM), 3π
rho	$\rho^0, \rho^\pm$	same as π	~ 775	4×10^{-27} s	1^-	$\rho \to \pi\pi$
	ω	$u\bar{u} + d\bar{d}$	782	10^{-24} s	1^-	$\omega \to \pi^+\pi^-\pi^0$
	$K^{*\pm}, K^{*0}, \bar{K}^{*0}$	same as K	~ 892	$\sim 10^{-26}$ s	1^-	$K\pi$ (strong)
	η'	$\sim u\bar{u} + d\bar{d} + s\bar{s}$	958	3×10^{-24} s	0^-	$\rho^0\gamma$, $\eta\pi\pi$
	ϕ	$s\bar{s}$	1020	1.5×10^{-25} s	1^-	$\phi \to K\bar{K}$, $\rho\pi$

Baryons:

	p	uud	938	$\gtrsim 10^{33}$ yrs	$\frac{1}{2}^+$	—
	n	udd	939	880 s	$\frac{1}{2}^+$	$n \to p e \bar{\nu}_e$
	Λ	uds	1116	2.6×10^{-10}	$\frac{1}{2}^+$	$p\pi^-$, $n\pi^0$ (w)
	Σ^+	uus	1189	0.8×10^{-10}	$\frac{1}{2}^+$	$p\pi^0$, $n\pi^+$ (weak)
	Σ^0	uds	1193	7×10^{-20}	$\frac{1}{2}^+$	$\Lambda\gamma$ (EM)
	Σ^-	dds	1197	1.5×10^{-10}	$\frac{1}{2}^+$	$n\pi^-$ (Weak)
	Δ^{++}	uuu				
	Δ^+	uud	~ 1232	$\sim 10^{-27}$ s	$\frac{3}{2}^+$	$n\pi$, $p\pi$ (strong)
	Δ^0	udd				
	Δ^-	ddd				
"cascade"	Ξ^0	uss	1314	3×10^{-10} s	$\frac{1}{2}^+$	$\Lambda\pi^0$ (weak)
	Ξ^-	dss	1321	1.6×10^{-10}	$\frac{1}{2}^+$	$\Lambda\pi^-$ (weak)
	$\Sigma^{*+}, \Sigma^{*0}, \Sigma^{*-}$	same as Σ	~ 1385	$\sim 10^{-26}$ s	$\frac{3}{2}^+$	$\Lambda\pi$, $\Sigma\pi$ (strong)
	Ξ^{*0}, Ξ^{*-}	same as Ξ	~ 1530	$\sim 7 \times 10^{-26}$ s	$\frac{3}{2}^+$	$\Xi\pi$ (strong)
	Ω^-	sss	1672	8×10^{-11} s	$\frac{3}{2}^+$	ΛK^-, $\Xi^0\pi^-$ (weak)

+ many other states...

Each particle is an energy level of the Hamiltonian

Hints toward an organizing principle:

(1) Some hadrons (e.g. kaons, Λ, $\Sigma^\pm$, Ξ, Ω^-), although they could be produced in collisions very easily, had decays much longer than other states.

(Nature of s quarks & weak interaction unknown.)
$\Rightarrow$ New additive quantum number "strangeness" S that is conserved by strong interaction, but must be violated in the decay (weak interaction).
The convention was that each s quark gives strangeness $S=-1$.
e.g. $S(K^+) = 1$, $S(\Lambda) = -1$, $S(\Omega^-) = -3$.

(2) Hadron states come in nearly degenerate multiplets:

e.g. $\begin{pmatrix} p \\ n \end{pmatrix}$, $\begin{pmatrix} \pi^+ \\ \pi^0 \\ \pi^- \end{pmatrix}$, Λ, $\begin{pmatrix} \Delta^{++} \\ \Delta^+ \\ \Delta^0 \\ \Delta^- \end{pmatrix}$, etc.

doublet ($I=\frac{1}{2}$) triplet ($I=1$) singlet ($I=0$) quadruplet ($I=\frac{3}{2}$)

Points to a symmetry called "isospin", rotation between components.
Different components have different electric charges & (possibly) decays $\Rightarrow$ EM and weak interaction violate this symmetry.
However, the strong interaction Hamiltonian should preserve this symmetry, since it is generating the masses.

That is, multiplets are representations of $SU(2)$ symmetry.

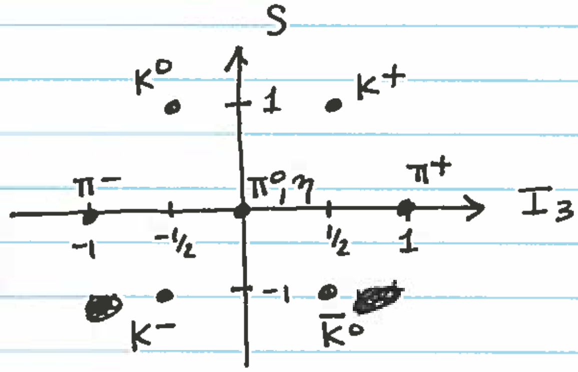
e.g. $I_3|p\rangle = +\frac{1}{2}|p\rangle$, $I_3|n\rangle = -\frac{1}{2}|n\rangle$
$|\vec{I}|^2|p\rangle = (I(I+1))|p\rangle = \frac{3}{4}|p\rangle$, same for $|n\rangle$

Works just like spin.

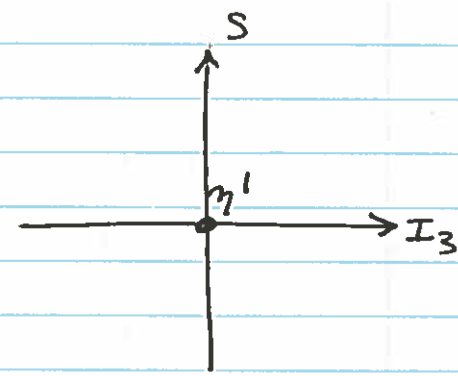
Eight fold Way: Gell-mann & Ne'eman showed how to embed these two approximate symmetries within a larger symmetry group ($SU(3)$).

First, consider plotting states of a given J^P in the $I_3 - S$ plane.

Pseudoscalar mesons:

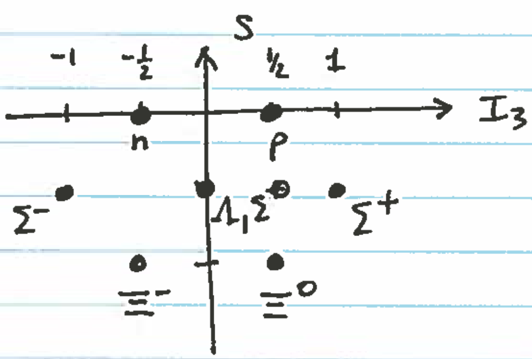


pseudoscalar octet



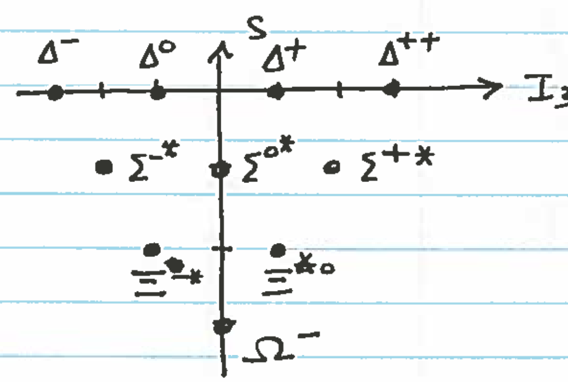
Singlet

spin-½ baryons



baryon octet

spin-3/2 baryons



baryon decuplet.

These patterns follow representations of $SU(3)$ symmetry.

Symmetry of the strong interaction:

Starting from the SM Lagrangian it is easy to see where the SU(3) symmetry comes from. Neglecting electroweak interactions, the quark Lagrangian is:

$$\mathcal{L} = \sum_q \bar{q}\left(i(\not{\partial} - i g_s T^A \not{G}^A) - m_q\right) q - \frac{1}{2} \mathrm{Tr}[G_{\mu\nu} G^{\mu\nu}]$$

sum over all quark fields u, d, s, c, b, t.
G^A_μ is gluon field.
m_q = quark mass for quark q.
$T^A = SU(3)_c$ generator $= \frac{\lambda^A}{2}$

For light quark hadrons, the hadron mass comes predominantly from strong interactions. Those quarks with masses $m_q \ll \Lambda_{QCD}$ can be taken to be approximately degenerate (or massless).

Considering only u, d, s quarks; and setting $m_{u,d,s} \to m_0$:

$$\mathcal{L} = \bar{q}\left(i(\not{\partial} - i g_s T^A) - m_0\right) q - \frac{1}{2} \mathrm{Tr}[G_{\mu\nu} G^{\mu\nu}]$$

where $q = \begin{pmatrix} u \\ d \\ s \end{pmatrix}$, triplet of quark flavors.

$\mathcal{L}$ has SU(3) flavor symmetry: $q \to Uq$, where U is SU(3) matrix.

$$\bar{q}\left(i(\not{\partial} - i g_s T^A \not{G}^A) - m_0\right) q \to \bar{q} U^\dagger \left(i(\not{\partial} - i g T^A \not{G}^A) - m_0\right) U q$$
$$= \bar{q}\left(i(\not{\partial} - i g T^A \not{G}^A) - m_0\right) q$$

Can also write e.g. EM interactions as

$$\mathcal{L}_{int} = -e \bar{q} Q \gamma^\mu q A_\mu \quad \text{where } Q = \begin{pmatrix} 2/3 & & 0 \\ & -1/3 & \\ 0 & & -1/3 \end{pmatrix}$$

Not invariant under SU(3) flavor $\Rightarrow$ EM interactions violate symmetry.

Note: m_s is comparable to Λ_{QCD} ($m_s \sim 100$ MeV).
So m_s provides a non-negligible breaking of SU(3)flavor
SU(2) subgroup acting on $\binom{u}{d}$ should be a much better symmetry than SU(3)flavor by itself. This is isospin symmetry.

Young tableaux & Irreducible representations:

Single quark $q = \begin{pmatrix} u \\ d \\ s \end{pmatrix}$ is fundamental rep of SU(3). "3"
i.e. 3-dim. representation. $q_i \to U_{ij} q_j$

Single antiquark $\bar{q} = \begin{pmatrix} \bar{u} \\ \bar{d} \\ \bar{s} \end{pmatrix}$ is antifundamental rep "$\bar{3}$"
also 3-dim. $\bar{q}_i \to U^*_{ij} \bar{q}_j$

~~Two quark state~~

(diquark)

Now consider how a two-quark state transforms:
Assume for now quarks are distinguishable, labeled (1), (2).

$$q^{(1)}_i q^{(2)}_j \to U_{ik} U_{jl} q^{(1)}_k q^{(2)}_l$$

Labels (1,2) assume quarks are distinguishable, kind of artificial.

There are 9 possible states dim = 9.

This state is the product of two 3's: 3×3. But the 3×3 representation is reducible — can be decomposed into lower dimensional representations that don't mix under SU(3).

Symmetric rep: $q^{(1)}_i q^{(2)}_j + q^{(1)}_j q^{(2)}_i \to U_{ik} U_{jl} (q^{(1)}_k q^{(2)}_l + q^{(1)}_l q^{(2)}_k)$
6-dimensional rep. (6 possible i,j)

antisymmetric rep: $q^{(1)}_i q^{(2)}_j - q^{(1)}_j q^{(2)}_i \to U_{ik} U_{jl} (q^{(1)}_k q^{(2)}_l - q^{(1)}_l q^{(2)}_k)$
3-dimensional rep. (3 possible i,j)

Write antisym rep in compact notation:

$$\varepsilon_{ijk}\, q_j^{(1)} q_k^{(2)} \rightarrow \varepsilon_{ijk}\, U_{jm} U_{kn}\, q_m^{(1)} q_n^{(2)}$$

Note: ε_{ijk} is invariant under $SU(3)$:

$$\varepsilon_{ijk}\, U_{il} U_{jm} U_{kn} = \varepsilon_{lmn}$$

$$\Rightarrow \quad \varepsilon_{ijk}\, U_{jm} U_{kn} = U_{il}^{*}\, \varepsilon_{lmn}$$

So antisym rep transforms as a $\bar{3}$.

$$\varepsilon_{ijk}\, q_j^{(1)} q_k^{(2)} \rightarrow U_{il}^{*}\, \varepsilon_{lmn}\, q_m^{(1)} q_n^{(2)}$$

So we decompose a diquark as:

$$3 \times 3 = 6 + \bar{3}$$

We can represent this graphically using <u>Young tableaux</u>.

$\boxed{i}$ = fundamental rep. e.g. with index i

$\boxed{j}$ = fundamental rep. with index j.

$\begin{array}{|c|}\hline i \\ \hline j \\ \hline \end{array}$ = i,j contracted antisymmetrically (3)

$\begin{array}{|c|c|}\hline i & j \\ \hline \end{array}$ = i,j contracted symmetrically. (6)

Use Young tableaux to find irreducible representations:
Join together boxes in all possible ways subject to following rules:

1. length of rows does not increase from top to bottom
2. length of columns does not increase from left to right.
3. No more than 3 boxes in a column for $SU(3)$ (or N for $SU(N)$)

diquark: $\square \times \square = \begin{array}{|c|}\hline \ \\ \hline \ \\ \hline\end{array} + \begin{array}{|c|c|}\hline \ & \ \\ \hline\end{array}$

Counting the dimension of a rep: number of ways put indices 1, 2, 3 into each box subject to rules:

(1) indices, in a row, do not decrease from left to right (sym. indices)

(2) indices, in a column, increase from top to bottom. (antisym. indices)

$\square = \boxed{1}, \boxed{2}, \boxed{3}$ 3-dim

$\begin{array}{|c|}\hline \ \\ \hline \ \\ \hline\end{array} = \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline\end{array}, \begin{array}{|c|}\hline 1 \\ \hline 3 \\ \hline\end{array}, \begin{array}{|c|}\hline 2 \\ \hline 3 \\ \hline\end{array}$ 3-dim

$\begin{array}{|c|c|}\hline \ & \ \\ \hline\end{array} = \begin{array}{|c|c|}\hline 1 & 1 \\ \hline\end{array}, \begin{array}{|c|c|}\hline 1 & 2 \\ \hline\end{array}, \begin{array}{|c|c|}\hline 1 & 3 \\ \hline\end{array}, \begin{array}{|c|c|}\hline 2 & 2 \\ \hline\end{array}, \begin{array}{|c|c|}\hline 2 & 3 \\ \hline\end{array}, \begin{array}{|c|c|}\hline 3 & 3 \\ \hline\end{array}$ 6-dim.

So $\square \times \square = \begin{array}{|c|}\hline \ \\ \hline \ \\ \hline\end{array} + \begin{array}{|c|c|}\hline \ & \ \\ \hline\end{array}$

$3 \times 3 = \bar{3} + 6$

meson ($q\bar{q}$): $\square \times \begin{array}{|c|}\hline \ \\ \hline \ \\ \hline\end{array} = \begin{array}{|c|}\hline \ \\ \hline \ \\ \hline \ \\ \hline\end{array} + \begin{array}{|c|c|}\hline \ & \ \\ \hline \ \\ \cline{1-1}\end{array}$

$3 \times \bar{3}$ $\quad$ 1 $\quad$ 8

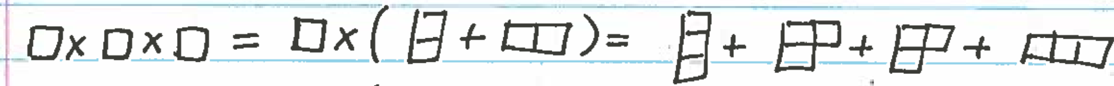
singlet $\quad$ octet

$\begin{array}{|c|}\hline \ \\ \hline \ \\ \hline \ \\ \hline\end{array} = \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline 3 \\ \hline\end{array}$ all indices antisym.

$\begin{array}{|c|c|}\hline \ & \ \\ \hline \ \\ \cline{1-1}\end{array} = \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 \\ \cline{1-1}\end{array}, \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 \\ \cline{1-1}\end{array}, \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array}, \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 3 \\ \cline{1-1}\end{array}, \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}, \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 3 \\ \cline{1-1}\end{array}, \begin{array}{|c|c|}\hline 2 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}, \begin{array}{|c|c|}\hline 2 & 3 \\ \hline 3 \\ \cline{1-1}\end{array}$

(Note: same reason we have 8 gluons — octet of $SU(3)_c$)

baryon : qqq



$3 \times 3 \times 3 = 3 \times (\bar{3} + 6) = 1 + 8 + 8 + 10$

singlet octet decuplet

Expect meson & baryon states to fall into representations of $SU(3)$. But we also need to consider spin, spatial, & color states.

Mesons: $q\bar{q}$ are distinguishable, so don't need to worry about having e.g. antisym. wavefunction.

Non-relativistic quark model: naive approx. meson as $q\bar{q}$ hydrogen-like bound state. Lowest lying mesons have $L=0$ spatial wave function. Spin of mesons due to spin of quarks.

We can use Young Tableaux to consider spin wave function (applied to $SU(2)$):

$\underbrace{2 \quad 2}_{\text{2 spin } \frac{1}{2} \text{ quarks}} \qquad 1 + 3$

spin singlet ($J=0$) spin triplet ($J=1$)

Lastly, the color wave function must be in the singlet state for $SU(3)_c$. (color neutral).

And also note: $q\bar{q}$ has intrinsic parity (-1), so the parity of the wavefunction is $-(-1)^L = -1$ since $L=0$.

Four possibilities:

pseudoscalar octet or singlet (both together = nonet)

vector octet or singlet

Octet rep. corresponds to matrix M_{ij} transforming $M_{ij} \to U_{ik} U^*_{j\ell} M_{k\ell}$ or in matrix form $M \to U M U^\dagger$. This is adjoint rep.
M represented by the 8 generators of $SU(3)$ flavor:

$$M = M^A T^A, \qquad T^A = \frac{\lambda^A}{2} \qquad M = \text{Hermitian traceless matrix.}$$

$$= \frac{1}{2}\begin{pmatrix} M^3 + M^8/\sqrt{3} & M^1 - iM^2 & M^4 - iM^5 \\ M^1 + iM^2 & -M^3 + M^8/\sqrt{3} & M^6 - iM^7 \\ M^4 + iM^5 & M^6 + iM^7 & -2/\sqrt{3}\, M^8 \end{pmatrix}$$

Each M^A is a degree of freedom, but useful to define particular linear combinations as follows: for the pseudoscalar octet

$$M = \begin{pmatrix} \frac{\pi^0}{\sqrt{2}} + \frac{\eta}{\sqrt{6}} & \pi^+ & K^+ \\ \pi^- & -\frac{\pi^0}{\sqrt{2}} + \frac{\eta}{\sqrt{6}} & K^0 \\ K^- & \bar{K}^0 & -\frac{2\eta}{\sqrt{6}} \end{pmatrix} \qquad \text{i.e. } \pi^\pm = \frac{M^1 \mp iM^2}{2}, \text{ etc.}$$

Recall for $SU(2)$, there is one diagonal generator T^3. Each state of a given $SU(2)$ multiplet is labeled by T^3. (I.e. each spin component is labeled by S_z.)

For $SU(3)$, there are two diagonal generators T^3 & T^8. Since $SU(3)_{\text{flavor}}$ is a symmetry of the strong Hamiltonian, states can be labeled by their eigenvalues under T^3, T^8.

e.g. K^0
$$\left[T^3, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & K^0 \\ 0 & 0 & 0 \end{pmatrix}\right] = \left[\frac{1}{2}\begin{pmatrix} 1 & & \\ & -1 & \\ & & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & K^0 \\ 0 & 0 & 0 \end{pmatrix}\right] = -\frac{1}{2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & K^0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\left[T^8, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & K^0 \\ 0 & 0 & 0 \end{pmatrix}\right] = \left[\frac{1}{\sqrt{3}}\begin{pmatrix} 1 & & \\ & 1 & \\ & & -2 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & K^0 \\ 0 & 0 & 0 \end{pmatrix}\right] = \frac{1}{\sqrt{3}}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & K^0 \\ 0 & 0 & 0 \end{pmatrix}$$

How do we get isospin & strangeness?

isospin symmetry is the SU(2) subgroup of SU(3) flavor acting on $\binom{u}{d}$.

$$U = \begin{pmatrix} V & 0 \\ 0 & 1 \end{pmatrix} \quad , \quad V = \text{SU(2) isospin transform.}$$

↑ 3x3 SU(3) transform

Acting on M we have:

$$M \to UMU^\dagger = \begin{pmatrix} V & 0 \\ 0 & 1 \end{pmatrix} M \begin{pmatrix} V^\dagger & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \left(\begin{array}{c|c} V\begin{pmatrix} \pi^0/\sqrt{2} & \pi^+ \\ \pi^- & -\pi^0/\sqrt{2} \end{pmatrix} V^\dagger + \frac{\eta}{\sqrt{6}} \mathbb{1} & V\begin{pmatrix} K^+ \\ K^0 \end{pmatrix} \\ \hline (K^-, \bar{K}^0) V^\dagger & -\frac{2\eta}{\sqrt{6}} \end{array}\right)$$

$\pi^\pm, \pi^0$ = isospin triplet (adjoint rep of SU(2))

η = isospin singlet

$\begin{pmatrix} K^+ \\ K^0 \end{pmatrix}$ = isospin doublet

$\begin{pmatrix} \bar{K}^0 \\ K^- \end{pmatrix}$ = isospin doublet

What we called I_3 before is just T_3.

Before we consider strangeness, note there is a third quantum number, baryon number. In SU(3) flavor space, the baryon number operator is proportional to $\mathbb{1} = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}$ since SU(3) multiplet components each have same B.

Strangeness $S = \frac{1}{\sqrt{3}} T^8 - B$.

B = 0 for mesons, but formula applies to baryon multiplets also.

State:	I_3	I	T^8	S
π^+	1	1	0	0
π^0	0	1	0	0
π^-	-1	1	0	0
η	0	0	0	0
K^+	1/2	1/2	$\sqrt{3}$	1
K^0	-1/2	1/2	$\sqrt{3}$	1
$\bar{K}^0$	1/2	1/2	$-\sqrt{3}$	-1
K^-	-1/2	1/2	$-\sqrt{3}$	-1
η'	0	0	0	0

Same as what we plotted in I_3–S plane

η': Singlet.

Now we can write down a Lagrangian for these states:

$$\mathcal{L} = \frac{1}{2}\mathrm{Tr}[\partial_\mu M \partial^\mu M] - \frac{1}{2}\mu_8^2 \mathrm{Tr}[M^2] + \mathcal{L}_{int}$$
$$+ \frac{1}{2}\partial_\mu \eta' \partial^\mu \eta' - \frac{1}{2}\mu_1^2 \eta'^2$$

Just focus on free Lagrangian. Numerical coefficients chosen such that meson fields in canonical form:

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \pi^0)^2 - \frac{1}{2}\mu_8^2 \pi^{0\,2} + (\partial_\mu \pi^+)(\partial^\mu \pi^-) - \mu_8^2 \pi^+ \pi^-$$
$$+ \frac{1}{2}(\partial_\mu \eta)^2 - \frac{1}{2}\mu_8^2 \eta^2 + \ldots$$
$$+ \frac{1}{2}(\partial_\mu \eta')^2 - \frac{1}{2}\mu_1^2 \eta'^2 + \mathcal{L}_{int}.$$

Consequence of SU(3) flavor: all octet states have common mass μ_8, singlet state has mass μ_1.

$\Rightarrow \quad m_\pi^2 = m_\eta^2 = m_K^2 \qquad$ prediction from exact SU(3) flavor symmetry

However, the masses of the mesons don't satisfy this relation
$\Rightarrow$ SU(3)flavor is only an approximate symmetry.
One of the largest sources of SU(3)flavor-breaking is the strange quark mass $m_s \gg m_{u,d}$.

Let's go back to the original quark Lagrangian and put in m_s:

$$\mathcal{L} = \bar{q}\left(i\not{D} - \begin{pmatrix} m_0 & & \\ & m_0 & \\ & & m_s \end{pmatrix}\right) q + \ldots$$

$$= \bar{q}\left(i\not{D} - m_0 \mathbb{1} - \Delta_s\right) q + \ldots$$

where $\not{D} = \not{\partial} - i g_s T^A \not{G}^A$ (QCD only), $q = \begin{pmatrix} u \\ d \\ s \end{pmatrix}$

$m_0 = m_u = m_d$ common u, d mass

$$\Delta_s = \begin{pmatrix} 0 & & \\ & 0 & \\ & & m_s - m_0 \end{pmatrix} \approx \begin{pmatrix} 0 & & \\ & 0 & \\ & & m_s \end{pmatrix} \quad \text{since } m_s \gg m_0.$$

Obviously Δ_s breaks SU(3)flavor, picks out preferred component.
But we can restore SU(3)flavor by treating Δ_s as a spurion.

That is, assume Δ_s also transforms under SU(3)flavor as

$$\Delta_s \to U \Delta_s U^\dagger$$

Then $\bar{q}\Delta_s q \to \bar{q} U^\dagger U \Delta_s U^\dagger U q = \bar{q}\Delta_s q$ is invariant.
It doesn't really transform, this is just a trick.

Now we ~~can~~ must include Δ_s in the effective theory for the meson states.

$$\mathcal{L} = \tfrac{1}{2}\mathrm{Tr}[\partial_\mu M \partial^\mu M] - \tfrac{1}{2}\mu_8^2 \mathrm{Tr}[M^2] - \tfrac{1}{2}\alpha \mathrm{Tr}[\Delta_s M^2] - \tfrac{1}{2}\beta \eta' \mathrm{Tr}[\Delta_s M]$$

$$+ \tfrac{1}{2}(\partial_\mu \eta')^2 - \tfrac{1}{2}\mu_1^2 \eta'^2 + \mathcal{L}_{int}$$

$$- \tfrac{\gamma}{2}\mathrm{Tr}[\Delta_s]\eta'^2$$

We are allowed to write down two extra terms in the free $\mathcal{L}$ (working only at lowest order in m_s)

Expanding out $\mathcal{L}$:

$$\mathcal{L} = \ldots \text{ kinetic terms } - \frac{1}{2}\mu_8^2\left(\pi^{0\,2} + 2\pi^+\pi^-\right)$$

$$- \frac{1}{2}\left(\mu_8^2 + \frac{\alpha m_s}{2}\right)\left(2K^0\bar{K}^0 + 2K^+K^-\right)$$

$$- \frac{1}{2}(\eta, \eta')\begin{pmatrix} \mu_8^2 + \frac{2}{3}\alpha m_s & -\beta m_s/\sqrt{6} \\ -\frac{\beta m_s}{\sqrt{6}} & \mu_1^2 + \gamma m_s \end{pmatrix}\begin{pmatrix} \eta \\ \eta' \end{pmatrix}$$

Spurion technique tells us how symmetry breaking terms propagate from the quark theory to the hadron theory, even though we can compute coefficients of the hadron theory. Group theory restricts the possible numbers of terms & unknown parameters.

Strange quark has two important effects:

(1) Splits degeneracy between octet states.

(2) mixes η, η'. So the physical η, η' states are an admixture of the "octet η" and the "singlet η'"

But mixing angle is known to be small experimentally ($\sim 20°$)

Approximating $\beta \approx 0$, we have:

$$m_\pi^2 = \mu_8^2, \quad m_K^2 = \left(\mu_8^2 + \frac{\alpha m_s}{2}\right), \quad m_\eta^2 = \mu_8^2 + \frac{2}{3}\alpha m_s$$

$$\Rightarrow \quad 4m_K^2 = m_\pi^2 + 3m_\eta^2 \qquad \text{satisfied at } \sim 5\%!$$

We can do same exercise with baryon octet:

$$B = \begin{pmatrix} \frac{\Sigma^0}{\sqrt{2}} + \frac{\Lambda^0}{\sqrt{6}} & \Sigma^+ & p \\ \Sigma^- & -\frac{\Sigma^0}{\sqrt{2}} + \frac{\Lambda^0}{\sqrt{6}} & n \\ \Xi^- & \Xi^0 & -\sqrt{\frac{2}{3}}\Lambda^0 \end{pmatrix}$$

(No singlet baryon state due to antisymmetry of wave function.)

$$\mathcal{L} = \mathrm{Tr}[\bar{B}(i\not{D} - \mu_0)B] - \alpha\, \mathrm{Tr}[\bar{B}\Delta_s B] - \beta\, \mathrm{Tr}[\Delta_s \bar{B} B]$$

two possible terms allowed by symmetry. (note: $B^\dagger \neq B$)

Expanding out:

$$m_{p,n} = \mu_0 + \alpha m_s$$
$$m_\Lambda = \mu_0 + \tfrac{2}{3}\alpha m_s + \tfrac{2}{3}\beta m_s$$
$$m_\Sigma = \mu_0$$
$$m_\Xi = \mu_0 + \beta m_s$$

This implies:

$$\frac{m_\Sigma + 3m_\Lambda}{2} = \mu_0 + \alpha m_s + \beta m_s =$$

$$\frac{m_\Sigma + 3m_\Lambda}{2} = m_\Sigma + m_\Xi$$

This is the Gell-Mann-Okubo mass formula,
Works to better than 1%!