Non-abelian Group Theory and Non-abelian Gauge Theory Review

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1. Groups in Particle Physics

When neutrons were discovered, a few fun facts were noticed:

- Protons and neutrons almost had the same mass, and they behaved almost the same[1]. And thus we call them nucleons.
- In particular, the strong interaction between nucleons, either it is between two protons, two neutrons, or a proton and a neutron, is always the same.

By then, we had no powerful accelerators to see the substructure of the nucleons and we believed them as fundamental and the strong interactions were transmitted by pions. There are three different pions: π^+, π^-, and π^0. By the knowledge of the fun facts, it is reasonable to assume that the action for these particles should be invariant under a global $SU(2)$ symmetry, historically known as the isospin symmetry.

a) By dimensions of the representations of $SU(2)$, which representation are the nucleons and pions in respectively?

   The n fields transform as an $n \times n$ dimensional representation L if

$$\left[T^i, \Phi_a(x)\right] = -L^i_{ab}\Phi_b(x).$$

b) Using dimensional analysis, write down all the possible relevant terms that could exist in the Lagrangian that respect the $SU(2)$ symmetry.

c) Show that each term you write down is indeed $SU(2)$ invariant by checking

$$\left[T^i, \text{your term here}\right] = 0$$

   using the representations for fundamentals and adjoints in $SU(2)$[2],

$$L^i_f = \frac{\tau^i}{2},\ L^i_{adj,jk} = -i\epsilon_{ijk}$$

[1] Except proton has a positive electric charge while the neutron is neutral.
[2] Only the interaction term will actually use the specific representation.

d) **Yukawa's Theory of Mesons:** In the Yukawa model of nucleons and pions the $SU(2)$ (isospin) invariant Lagrangian density is

$$\mathcal{L}_0 = \bar{\psi}\left(i\,\partial\!\!\!/ - mI + ig_\pi \gamma^5 \vec{\pi}\cdot\vec{\tau}\right)\psi + \frac{1}{2}\left[(\partial_\mu \vec{\pi})^2 - \mu^2 \vec{\pi}^{\,2}\right] - \lambda\left(\vec{\pi}^{\,2}\right)^2,$$

where

$$\psi = \begin{pmatrix} \psi_p \\ \psi_n \end{pmatrix}, \qquad \pi = \begin{pmatrix} \pi_1 \\ \pi_2 \\ \pi_3 \end{pmatrix}, \qquad \pi^\pm = \frac{\pi_1 \mp i\pi_2}{\sqrt{2}}, \qquad \pi^0 = \pi_3.$$

Note that the field ψ has two kinds of indices – the usual spinor index as well as an $SU(2)$ index. The nucleon and pion fields transform under the fundamental and adjoint representations, respectively.

Show that the vertices for $\bar{p}p\pi^0, \bar{n}n\pi^0, \bar{n}p\pi^-$, and $\bar{p}n\pi^+$ (given by the relevant coefficients in $i\mathcal{L}_0$) are related and given by $-g_\pi\gamma^5, +g_\pi\gamma^5, -\sqrt{2}g_\pi\gamma^5, -\sqrt{2}g_\pi\gamma^5$, respectively.

2. Quark Scattering and QCD Potential

Quarks are fermions that live in the fundamental representation of $SU(3)$. Quarks come in six flavors (up, down, charm, strange, top, and bottom), and are constituents of hadrons such as protons and neutrons. Strong (aka QCD or $SU(3)$) interactions conserve flavor. Due to confinement, the initial and final states of any scattering experiment only contains bound states of quarks instead of isolated quarks. Let us oversimplify our world by considering scattering of the quarks instead of hadrons. We will also ignore electroweak interactions to further simplify things.

Consider the process $q_i^r \bar{q}_j^r \to q_k^s \bar{q}_l^s$ (assuming they carry momentum $p_1^\mu, p_2^\mu, p_3^\mu, p_4^\mu$ respectively), where r, s are flavor indices and i, j, k, l are color indices.

a) Write down the Lagrangian and determine the Feynman rule(s) for the quark-gluon vertex or vertices.

b) Draw the Feynman diagram(s) that contribute to this process at leading order. Note that depending on the final states, there are two different scenarios.

c) Before we dive in and start calculating the amplitude, can you think of any Feynman diagram(s) in QED that look similar to the ones you just drew? Draw the corresponding QED diagrams.

d) For each corresponding scenario in QED, write down the amplitude of the diagram according to the Feynman rules.

e) For each of the two quark scattering scenarios, write down the amplitude of the diagram according to the Feynman rules.

f) Compare each scenario of quark scattering and its QED pair and comment on your observation.

g) For each scenario of the quark scattering, write down the amplitude squared, averaged over initial colors and spins and summed over final colors and spins. Perform the color sums only. *Hint*:

$T^aT^bT^a = -\frac{1}{6}T^b$, $\mathrm{Tr}[T^aT^b] = \frac{1}{2}\delta^{ab}$ where T^a are the generators of $SU(3)$ in the fundamental representation.

h) Consult the appendix to determine the amplitude squared, averaged over initial colors and spins and summed over final colors and spins, for both scenarios.

i) *Optional*: If you completed the optional problem 2 of QFT I Tutorial 4, you know that the quantum field theory amplitude $i\mathcal{M}$ for two particles to scatter from momentum states $\pm\mathbf{p}$ to momentum states $\pm\mathbf{p}'$ in the non-relativistic limit is related to the classical potential $V(r)$ by:

$$\mathcal{M} = -(2m)^2 \int d^3r V(\mathbf{r}) e^{-i(\mathbf{p}-\mathbf{p}')\cdot\mathbf{r}} \tag{1}$$

where m is the mass of the particles. By considering the nonrelativistic limit of electron-positron scattering, derive the Coulomb potential $V(r) = -\frac{\alpha}{r}$ from QED.

j) Apart from group theoretic factors, the same calculation gives the QCD potential at short distances where perturbation theory works. Assuming the results of the previous part, show that the perturbative term in the QCD potential is $V_{\mathrm{QCD}} = -\frac{4}{3}\frac{\alpha_s}{r}$.

k) Which diagrams contribute to $q_i^r \bar{q}_j^r \to q_k^s \bar{q}_l^s$ at one loop? Which are UV-divergent? Which are IR-divergent?

A. Some Known Spin-Averaged Amplitudes-Squared

You may find the following QED results that you calculated in QFT I useful.

a) The process of $(e^-e^+ \to \mu^-\mu^+)$

$$\frac{1}{4}\sum |\bar{M}_{fi}|^2 = 2e^4 \frac{(t^2+u^2)}{s^2}.$$

b) The process of $(e^-e^+ \to e^-e^+)$

$$\frac{1}{4}\sum |\bar{M}_{fi}|^2_{Bhabha} = 2e^4 \left[\frac{t^2+u^2}{s^2} + \frac{s^2+u^2}{t^2} + \frac{2u^2}{st}\right]$$

Recall that the Mandelstam variables are defined by

$$\begin{aligned} s &\equiv (p_1+p_2)^2 = (p_3+p_4)^2 \\ t &\equiv (p_1-p_3)^2 = (p_2-p_4)^2 \\ u &\equiv (p_1-p_4)^2 = (p_2-p_3)^2 \end{aligned} \tag{2}$$