Non-abelian Group Theory and Non-abelian Gauge Theory Review

5 January 2016

1. Groups in Particle Physics

When neutrons were discovered, a few fun facts were noticed:

- Protons and neutrons almost had the same mass, and they behaved almost the same[1]. And thus we call them nucleons.
- In particular, the strong interaction between nucleons, either it is between two protons, two neutrons, or a proton and a neutron, is always the same.

By then, we had no powerful accelerators to see the substructure of the nucleons and we believed them as fundamental and the strong interactions were transmitted by pions. There are three different pions: π^+, π^-, and π^0. By the knowledge of the fun facts, it is reasonable to assume that the action for these particles should be invariant under a global $SU(2)$ symmetry, historically known as the isospin symmetry.

a) By dimensions of the representations of $SU(2)$, which representation are the nucleons and pions in respectively?

   The n fields transform as an $n \times n$ dimensional representation L if

$$\left[T^i, \Phi_a(x)\right] = -(L^i \Phi(x))_a.$$

b) Using dimensional analysis, write down all the possible relevant terms that could exist in the Lagrangian that respect the $SU(2)$ symmetry.

c) Show that each term you write down is indeed $SU(2)$ invariant by checking

$$\left[T^i, \text{your term here}\right] = 0$$

   using the representations for fundamentals and adjoints in $SU(2)$[2],

$$L_f^i = \frac{\tau^i}{2},\ L^i_{adj,jk} = -i\epsilon_{ijk}$$

[1] Except proton has a positive electric charge while the neutron is neutral.
[2] Only the interaction term will actually use the specific representation.

d) **Yukawa's Theory of Mesons:** In the Yukawa model of nucleons and pions the $SU(2)$ (isospin) invariant Lagrangian density is

$$\mathcal{L}_0 = \bar{\psi}\left(i\,\partial\!\!\!/ - mI + ig_\pi\gamma^5\vec{\pi}\cdot\vec{\tau}\right)\psi + \frac{1}{2}\left[(\partial_\mu\vec{\pi})^2 - \mu^2\vec{\pi}^{\,2}\right] - \lambda\left(\vec{\pi}^{\,2}\right)^2,$$

where

$$\psi = \begin{pmatrix}\psi_p\\ \psi_n\end{pmatrix}, \qquad \pi = \begin{pmatrix}\pi_1\\ \pi_2\\ \pi_3\end{pmatrix}, \qquad \pi^{\pm} = \frac{\pi_1 \mp i\pi_2}{\sqrt{2}}, \qquad \pi^0 = \pi_3.$$

Note that the field ψ has two kinds of indices – the usual spinor index as well as an $SU(2)$ index. The nucleon and pion fields transform under the fundamental and adjoint representations, respectively.

Show that the vertices for $\bar{p}p\pi^0, \bar{n}n\pi^0, \bar{n}p\pi^-$, and $\bar{p}n\pi^+$ (given by the relevant coefficients in $i\mathcal{L}_0$) are related and given by $-g_\pi\gamma^5, +g_\pi\gamma^5, -\sqrt{2}g_\pi\gamma^5, -\sqrt{2}g_\pi\gamma^5$, respectively.

Solution:

a) Since there are two types of nucleons and three types of pions. The nucleons will be in the fundamental representation and the pions will be in the adjoint representation.

b) The nucleons are fermions, so the dimension is $\frac{3}{2}$. The pions are bosons, so the dimension is 1. The renormalizable terms will can write down will involve the kinetic and mass term of the nucleons and the pions, the self-interaction of the pions,

$$\bar{\psi}\,\partial\!\!\!/\psi, m\bar{\psi}\psi, (\partial_\mu\vec{\pi})^2, \mu^2\vec{\pi}^{\,2}, \lambda\left(\vec{\pi}^{\,2}\right)^2,$$

and there is only one term that involves both nucleons and pions and is renormalizable,

$$\bar{\psi}(\vec{\pi}\cdot\vec{\tau})\psi$$

c) As Dirac matrices do not affect the $SU(2)$ symmetry, we only need to show

$$[T^i, \psi^\dagger\psi] = 0$$

By definition, we have

$$[T^i, \psi_a] = -(L^i\psi)_a$$

Take the adjoint, we have

$$[T^i, \psi_a]^\dagger = -(\psi^\dagger L^i)_a = [\psi_a^\dagger, T^i] = -[T^i, \psi_a^\dagger]$$

thus we have

$$[T^i, \psi_a^\dagger] = (\psi^\dagger L^i)_a$$

Now use the following relationship regarding commutators,

$$[A, BC] = [A, B]C + B[A, C]$$

we have

$$[T^i, \psi_a^\dagger \psi_a] = (\psi^\dagger L^i)_a \psi_a + \psi_a^\dagger(-(L^i\psi)_a) = 0$$

The above proof does not depend on which representation the field lives in, so it would work for $\vec{\pi}^{\,2}$ too. The only other thing we need to handle is,

$$\psi^\dagger(\vec{\pi}\cdot\vec{\tau})\psi$$

We need to use

$$[A, BCD] = [A, B]CD + B[A, C]D + BC[A, D]$$

we have

$$\begin{aligned}
[T^i, \psi^\dagger(\vec{\pi}\cdot\vec{\tau})\psi] &= (\psi^\dagger L^i)(\vec{\pi}\cdot\vec{\tau})\psi + \psi^\dagger(-L^i_\pi\vec{\pi})\cdot\vec{\tau}\psi + \psi^\dagger(\vec{\pi}\cdot\vec{\tau})(-(L^i\psi)) &(1)\\
&= (\psi^\dagger L^i)(\vec{\pi}\cdot\vec{\tau})\psi + \psi^\dagger[(\vec{\pi}\cdot\vec{\tau}), L^i]\psi + \psi^\dagger(\vec{\pi}\cdot\vec{\tau})(-(L^i\psi)) = 0 &(2)
\end{aligned}$$

where we used

$$(-L^i_\pi\vec{\pi})\cdot\vec{\tau} = [(\vec{\pi}\cdot\vec{\tau}), L^i]$$

This can be proved using the specific representation of the fields,

$$(-L^i_\pi\vec{\pi})\cdot\vec{\tau} = -L^i_{\pi,kj}\pi^j\tau^k = i\epsilon_{ikj}\pi^j\tau^k$$

$$\begin{aligned}
[(\vec{\pi}\cdot\vec{\tau}), L^i] &= \left[\pi^j\tau^j, \frac{\tau^i}{2}\right] &(3)\\
&= \pi^j i\epsilon_{jik}\tau^k &(4)\\
&= i\epsilon_{ikj}\pi^j\tau^k &(5)\\
&= (-L^i_\pi\vec{\pi})\cdot\vec{\tau} &(6)
\end{aligned}$$

d) First we need to expand the g_π term in the Lagrangian density:

$$\begin{aligned}
\bar{\psi} i g_\pi \gamma^5 \vec{\pi}\cdot\vec{\tau}\psi &= i g_\pi \begin{pmatrix}\bar{\psi}_p & \bar{\psi}_n\end{pmatrix}\gamma^5 \begin{pmatrix}\pi_3 & \pi_1 - i\pi_2\\ \pi_1 + i\pi_2 & -\pi_3\end{pmatrix}\begin{pmatrix}\psi_p\\ \psi_n\end{pmatrix}\\
&= i g_\pi \begin{pmatrix}\bar{\psi}_p & \bar{\psi}_n\end{pmatrix}\gamma^5 \begin{pmatrix}\pi^0 & \sqrt{2}\pi^+\\ \sqrt{2}\pi^- & -\pi^0\end{pmatrix}\begin{pmatrix}\psi_p\\ \psi_n\end{pmatrix}\\
&= i g_\pi \begin{pmatrix}\bar{\psi}_p & \bar{\psi}_n\end{pmatrix}\gamma^5 \begin{pmatrix}\pi^0\psi_p + \sqrt{2}\pi^+\psi_n\\ \sqrt{2}\pi^-\psi_p - \pi^0\psi_n\end{pmatrix}\\
&= i g_\pi \left(\bar{\psi}_p\gamma^5\pi^0\psi_p + \sqrt{2}\bar{\psi}_p\gamma^5\pi^+\psi_n + \sqrt{2}\bar{\psi}_n\gamma^5\pi^-\psi_p - \bar{\psi}_n\gamma^5\pi^0\psi_n\right)
\end{aligned}$$

From the last line we can read off the Feynman rules, and see that the vertices for $\bar{p}p\pi^0, \bar{n}n\pi^0, \bar{n}p\pi^-$, and $\bar{p}n\pi^+$ are given respectively by $-g_\pi\gamma^5, +g_\pi\gamma^5, -\sqrt{2}g_\pi\gamma^5, -\sqrt{2}g_\pi\gamma^5$.

2. Quark Scattering

Quarks are fermions that live in the fundamental representation of $SU(3)$. Quarks come in six flavors (up, down, charm, strange, top, and bottom), and are constituents of hadrons such as protons and neutrons. Strong (aka QCD or $SU(3)$) interactions conserve flavor. Due to confinement, the initial and final states of any scattering experiment only contains bound states of quarks instead of isolated quarks. Let us oversimplify our world by considering scattering of the quarks instead of hadrons. We will also ignore electroweak interactions to further simplify things.

Consider the process $q_i^r \bar{q}_j^r \to q_k^s \bar{q}_l^s$ (assuming they carry momentum $p_1^\mu, p_2^\mu, p_3^\mu, p_4^\mu$ respectively), where r, s are flavor indices and i, j, k, l are color indices.

a) Write down the Lagrangian and determine the Feynman rule(s) for the quark-gluon vertex or vertices.

b) Draw the Feynman diagram(s) that contribute to this process at leading order. Note that depending on the final states, there are two different scenarios.

c) Before we dive in and start calculating the amplitude, can you think of any Feynman diagram(s) in QED that look similar to the ones you just drew? Draw the corresponding QED diagrams.

d) For each corresponding scenario in QED, write down the amplitude of the diagram according to the Feynman rules.

e) For each of the two quark scattering scenarios, write down the amplitude of the diagram according to the Feynman rules.

f) Compare each scenario of quark scattering and its QED pair and comment on your observation.

g) For each scenario of the quark scattering, write down the amplitude squared, averaged over initial colors and spins and summed over final colors and spins. Perform the color sums only. *Hint*: $T^a T^b T^a = -\frac{1}{6} T^b$, $\mathrm{Tr}[T^a T^b] = \frac{1}{2}\delta^{ab}$ where T^a are the generators of $SU(3)$ in the fundamental representation.

h) Consult the appendix to determine the amplitude squared, averaged over initial colors and spins and summed over final colors and spins, for both scenarios.

i) *Optional*: If you completed the optional problem 2 of QFT I Tutorial 4, you know that the quantum field theory amplitude $i\mathcal{M}$ for two particles to scatter from momentum states $\pm\mathbf{p}$ to momentum states $\pm\mathbf{p}'$ in the non-relativistic limit is related to the classical potential $V(r)$ by:

$$\mathcal{M} = -(2m)^2 \int d^3r V(\mathbf{r}) e^{-i(\mathbf{p}-\mathbf{p}')\cdot\mathbf{r}} \tag{7}$$

where m is the mass of the particles. By considering the nonrelativistic limit of electron-positron scattering, derive the Coulomb potential $V(r) = -\frac{\alpha}{r}$ from QED.

j) Apart from group theoretic factors, the same calculation gives the QCD potential at short distances where perturbation theory works. Assuming the results of the previous part, show that the perturbative term in the QCD potential is $V_{\mathrm{QCD}} = -\frac{4}{3}\frac{\alpha_s}{r}$.

k) Which diagrams contribute to $q_i^r \bar{q}_j^r \to q_k^s \bar{q}_l^s$ at one loop? Which are UV-divergent? Which are IR-divergent?

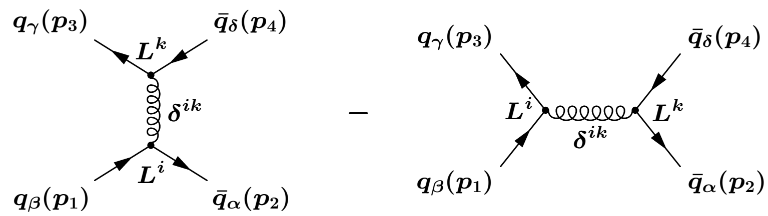
Solution:

a) The Lagrangian is just six fermions that live in the fundamental representation

$$\mathcal{L} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + \bar{\psi}_r \left(i \not{D} - m \right) \psi_r \tag{8}$$

since there is no flavor mixing in QCD. The vertex is rule is $-g\gamma^\mu \delta^{rs} T^a_{ij}$.

b) If the quarks in the final state have the same flavor as the quarks in the initial state (Scenario II), the Feynman diagram is given as follows, note the minus sign coming from the fact that



fermion fields are anti-commuting. If the quarks in the final state are in different flavor from the quarks in the initial state (Scenario I), the Feynman diagram will only consist the first diagram, as QCD cannot change flavor.

c) The Feynman diagram of the corresponding QED process is the process of Bhabha scattering ($e^- e^+ \to e^- e^+$) if the initial states and the final states are the same flavor. For the other scenario, the corresponding QED process is ($e^- e^+ \to \mu^- \mu^+$). We can obtain the diagrams by replacing the gluon lines with photon lines, the quark lines with lepton lines. Of course we have no color indices any more.

d) Scenario I (QED version)

$$i\mathcal{M}_{fi} = \frac{ie^2}{s} \left(\bar{u}_3 \gamma_\mu v_4 \ \bar{v}_2 \gamma^\mu u_1 \right). \tag{9}$$

Scenario II (QED version)

$$i\mathcal{M}_{fi} = ie^2 \left(\frac{\bar{u}_3 \gamma_\mu v_4 \ \bar{v}_2 \gamma^\mu u_1}{s} - \frac{\bar{u}_3 \gamma_\mu u_1 \ \bar{v}_2 \gamma^\mu v_4}{t} \right). \tag{10}$$

e) Scenario I

$$\begin{aligned} i\mathcal{M}_{fi} &= \left(i g_s \ \bar{u}_3 \gamma_\mu T^a_{kl} v_4 \right) \left(\frac{-i g^{\mu\rho} \delta^{ab}}{s} \right) \left(i g_s \ \bar{v}_2 \gamma_\rho T^b_{ji} u_1 \right) \\ &= \frac{i g_s^2}{s} T^a_{ji} T^a_{kl} \left(\bar{u}_3 \gamma_\mu v_4 \ \bar{v}_2 \gamma^\mu u_1 \right) \end{aligned} \tag{11}$$

Scenario II

$$
\begin{aligned}
i\mathcal{M}_{fi} &= (ig_s\ \bar{u}_3\gamma_\mu T^a_{kl}v_4)\left(\frac{-ig^{\mu\rho}\delta^{ab}}{s}\right)(ig_s\ \bar{v}_2\gamma_\rho T^b_{ji}u_1) \\
&- (ig_s\ \bar{u}_3\gamma_\mu T^a_{ki}u_1)\left(\frac{-ig^{\mu\rho}\delta^{ab}}{t}\right)(ig_s\ \bar{v}_2\gamma_\rho T^b_{jl}v_4) \\
&= ig_s^2\left(T^a_{ji}T^a_{kl}\frac{\bar{u}_3\gamma_\mu v_4\ \bar{v}_2\gamma^\mu u_1}{s} - T^a_{ki}T^a_{jl}\frac{\bar{u}_3\gamma_\mu u_1\ \bar{v}_2\gamma^\mu v_4}{t}\right).
\end{aligned}
\tag{12}
$$

f) By comparing the amplitudes, the only difference between the QCD version and the QED version is that QED version has no color factors and the coupling constant is e instead of g_s.

g) Scenario I. The amplitude squared is given by

$$
\begin{aligned}
\frac{1}{36}\sum_{\text{colors, spins}}|\mathcal{M}_{fi}|^2 &= \frac{1}{36}T^a_{kl}T^a_{ji}T^{*b}_{kl}T^{*b}_{ji}\sum_{\text{spins}}|\,(\bar{u}_3\gamma_\mu v_4\ \bar{v}_2\gamma^\mu u_1)\,|^2 \\
&= \frac{1}{36}\text{Tr}[T^aT^b]\text{Tr}[T^aT^b]\sum_{\text{spins}}|\,(\bar{u}_3\gamma_\mu v_4\ \bar{v}_2\gamma^\mu u_1)\,|^2 \\
&= \frac{1}{36}\frac{1}{2}\delta^{ab}\frac{1}{2}\delta^{ab}\sum_{\text{spins}}|\,(\bar{u}_3\gamma_\mu v_4\ \bar{v}_2\gamma^\mu u_1)\,|^2 \\
&= \frac{2}{9}\frac{g_s^4}{s^2}\frac{1}{4}\sum_{\text{spins}}|\,(\bar{u}_3\gamma_\mu v_4\ \bar{v}_2\gamma^\mu u_1)\,|^2
\end{aligned}
\tag{13}
$$

Scenairo II. The amplitude squared is given by three terms. The non cross terms will have the same color factor, but the cross term will have the following color factor

$$
\begin{aligned}
\left(\frac{1}{3}\right)^2 T^a_{ji}T^a_{kl}T^{*b}_{ki}T^{*b}_{jl} &= -\frac{1}{6}\left(\frac{1}{3}\right)^2 T^b_{jl}T^{*b}_{jl} \\
&= -\frac{1}{6}\left(\frac{1}{3}\right)^2 \text{Tr}(T^bT^b) \\
&= -\frac{2}{27}
\end{aligned}
\tag{14}
$$

h) Now we can just insert the color factors to the QED results to get the spin and color averaged amplitude-squared

Scenario I

$$
\frac{1}{36}\sum_{\text{colors, spins}}|\mathcal{M}_{fi}|^2 = \frac{4}{9}g_s^4\left(\frac{t^2+u^2}{s^2}\right) \tag{15}
$$

Scenario II

$$
\frac{1}{36}\sum_{\text{colors, spins}}|\mathcal{M}_{fi}|^2 = \frac{4}{9}g_s^4\left[\frac{t^2+u^2}{s^2}+\frac{s^2+u^2}{t^2}-\frac{2}{3}\frac{u^2}{st}\right] \tag{16}
$$

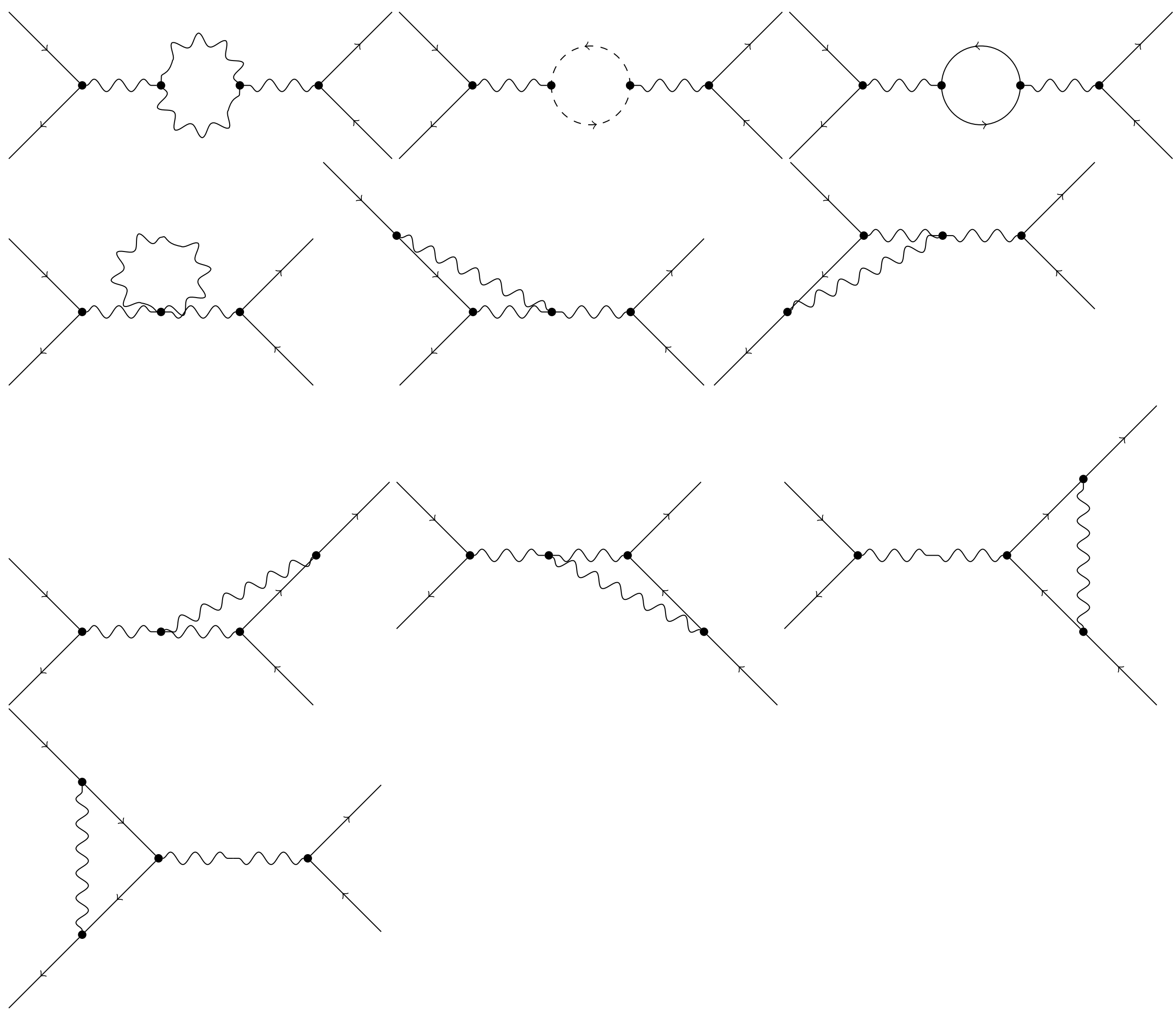
i) See Peskin and Schroeder page 125.

j) The difference between the QED and QCD case is that in the QCD case we should replace the factor of e^2 with

$$\begin{aligned} e^2 &\to \frac{1}{3} g_s^2 \sum_a T^a_{ji} T^a_{ij} \\ &= g_s^2 \frac{1}{3} \sum_a \frac{1}{2} \delta^{aa} \\ &= \frac{4}{3} g_s^2 \end{aligned} \tag{17}$$

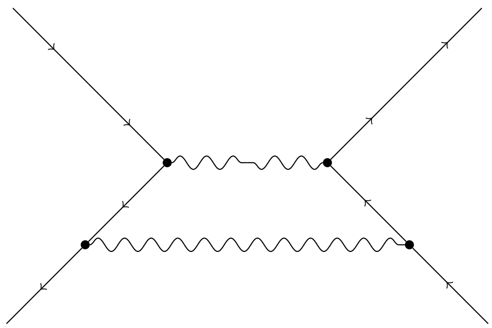
The factor of $\frac{1}{3}$ comes from color averaging. The indices on the generators have to be the same for a color-neutral state.

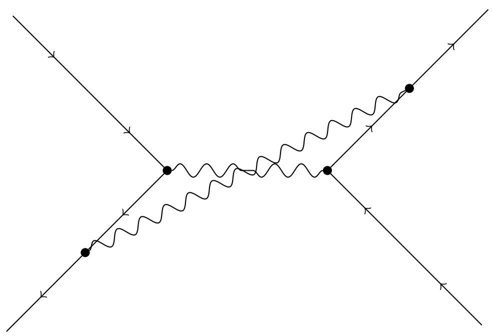
k) If the dotted line indicates the ghost, then the s-channel diagrams are:



If the quarks in the initial and final state are identical, then there is also a t-channel diagram for each of the above diagrams. The above diagrams are all naively both UV- and IR-divergent.

There are also the following diagrams:





plus additional similar diagrams if the quarks are identical. These diagrams are UV-finite, but IR-divergent.

A. Some Known Spin-Averaged Amplitudes-Squared

You may find the following QED results that you calculated in QFT I useful.

a) The process of $(e^- e^+ \to \mu^- \mu^+)$

$$\frac{1}{4}\sum |\bar{M}_{fi}|^2 = 2e^4 \frac{(t^2+u^2)}{s^2}.$$

b) The process of $(e^- e^+ \to e^- e^+)$

$$\frac{1}{4}\sum |\bar{M}_{fi}|^2_{Bhabha} = 2e^4 \left[\frac{t^2+u^2}{s^2} + \frac{s^2+u^2}{t^2} + \frac{2u^2}{st}\right]$$

Recall that the Mandelstam variables are defined by

$$\begin{aligned} s &\equiv (p_1+p_2)^2 = (p_3+p_4)^2 \\ t &\equiv (p_1-p_3)^2 = (p_2-p_4)^2 \\ u &\equiv (p_1-p_4)^2 = (p_2-p_3)^2 \end{aligned} \tag{18}$$