Higgs

7 January 2016

1 The Georgi-Glashow SU(2) Model

The Georgi-Glashow model was proposed as a theory of the weak interactions in 1972. It doesn't quite work, as you will see, but many key features of the electroweak theory are present. The Lagrangian for the Georgi-Glashow model is:

$$\mathcal{L}_{\text{GG}} = -\frac{1}{2}\text{Tr}(F_{\mu\nu}F^{\mu\nu}) + \text{Tr}\left[(D_\mu\phi)(D^\mu\phi)\right] + \mu^2\text{Tr}\phi^2 - \lambda(\text{Tr}\phi^2)^2 \tag{1}$$

The gauge group is $SU(2)$, and the scalar field $\phi = \phi^i T^i$ is a triplet (i.e. adjoint) whose components ϕ^i are real and

$$T^a = \frac{1}{2}\sigma^a \tag{2}$$

a) What values of ϕ minimize the potential?

b) By $SU(2)$ symmetry you can choose the vacuum to be $\langle\phi^1\rangle = \langle\phi^2\rangle = 0$, $\langle\phi^3\rangle = v$.

   Show that this choice of vacuum "breaks" two of the generators of $SU(2)$

$$e^{i\alpha_{1,2}T^{1,2}}\langle\phi\rangle e^{-i\alpha_{1,2}T^{1,2}} \neq \langle\phi\rangle \tag{3}$$

   and leaves one generator unbroken

$$e^{i\alpha_3 T^3}\langle\phi\rangle e^{-i\alpha_3 T^3} = \langle\phi\rangle \tag{4}$$

   by using the infinitesimal versions of equations (3) and (4).

c) Show that you can make a gauge choice (known as unitary gauge) so that $\phi^1 = \phi^2 = 0$, $\phi^3 = v + h$.

d) In unitary gauge, show that two of the three gauge bosons get a mass $M_A^2 = g^2v^2$, and that one of the gauge bosons remains massless.

The Georgi-Glashow model can accomodate the massive $W^\pm$ bosons as well as the photon, but it has no Z boson. The Z boson is necessary to explain weak neutral current interactions, which were first observed in elastic electron-neutrino scattering.

2 *The Higgs Sector*

This problem is essential for this course. The Lagrangian for the Higgs sector in the Standard Model is given by

$$\mathcal{L}_{\text{Higgs}} = (D_\mu H)^\dagger (D^\mu H) - V(H)$$

where H is an $SU(2)_L$ doublet of scalar fields and

$$V(H) = -\mu^2 H^\dagger H + \lambda (H^\dagger H)^2$$

is the $SU(2)_L \times U(1)_Y$ invariant potential.

Consider a constant value of H for which this potential is minimized. By $SU(2)$ symmetry we can always choose this value to be

$$\begin{pmatrix} 0 \\ \frac{v}{\sqrt{2}} \end{pmatrix} \tag{5}$$

where v is a constant. Recall that the $U(1)_Y$ charge of H is $\frac{1}{2}$.

a) Show that the generator of $SU(2)_L \times U(1)_Y$ that leaves the vacuum invariant is $T_3 + Y$.

Next consider small fluctuations of the Higgs field around this vacuum in the unitary gauge

$$H(x) = \begin{pmatrix} 0 \\ \frac{v+h(x)}{\sqrt{2}} \end{pmatrix}.$$

b) Show that the potential then takes on the form

$$V(h) = (\lambda v^2) h^2 + \lambda v h^3 + \frac{\lambda}{4} h^4.$$

c) Also show that the kinetic term for the Higgs boson in the Lagrangian becomes

$$\mathcal{L}_{\text{Kinetic}} = \frac{1}{2}(\partial_\mu h)(\partial^\mu h) + \frac{1}{8}\left(g' B_\mu - g X^3_\mu\right)\left(g' B^\mu - g X^{3\mu}\right)(v+h)^2 + \frac{1}{8}\left(g X^1_\mu - i g X^2_\mu\right)\left(g X^{1\mu} + i g X^{2\mu}\right)(v+h)^2$$

which shows that some of the vector bosons gain mass terms.

d) If we perform a field redefinition, we want the kinetic terms for the vector bosons to remain canonical normalized. Show that correct mass-eigenstate for the neutral particle above is

$$Z_\mu = \cos\theta_W X^3_\mu - \sin\theta_W B_\mu$$

where

$$\cos\theta_W = \frac{g}{\sqrt{(g')^2 + g^2}}$$

is the Weinberg angle.

e) Verify that under a gauge transformation involving $U(1)_{EM}$ ($U(1)_{EM}$ is the unbroken subgroup of $SU(2)_L \times U(1)_Y$), X^1_μ and X^2_μ transform into each other, and thus

$$W^{\pm\mu} = \frac{1}{\sqrt{2}}\left(X^1_\mu \mp iX^2_\mu\right)$$

represent charged particles under $U(1)_{EM}$.

f) Verify that the masses of the three massive bosons are given by

$$m_W^2 = \frac{1}{4}g^2v^2 \quad \text{and} \quad m_Z^2 = \frac{1}{4}v^2((g')^2 + g^2)$$

g) What is the expression for the vector field A_μ (given in terms of g, g', X^a_μ and B_μ) that remains massless? What is it called?

h) In general, the covariant derivative of field with $SU(2)_L \times U(1)_Y$ charge that is an $SU(2)$ doublet is given by

$$D_\mu = \partial_\mu - igX^a_\mu T^a - ig'YB_\mu \tag{6}$$

where

$$T^a = \frac{1}{2}\sigma^a \tag{7}$$

Please rewrite this covariant derivative in terms of $W^\pm_\mu$, Z_μ, A_μ and obtain the following form:

$$D_\mu = \partial_\mu - i\frac{g}{\sqrt{2}}(W^+_\mu T^+ + W^-_\mu T^-) - i\frac{g}{\cos\theta_W}Z_\mu(T^3 - Q\sin^2\theta_W) - ieA_\mu Q \tag{8}$$

where we also define

$$T^\pm \equiv T^1 \pm iT^2 \tag{9}$$

and

$$Q \equiv T_3 + Y,\ e \equiv \frac{gg'}{\sqrt{g^2 + g'^2}}$$