Higgs

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1 The Georgi-Glashow SU(2) Model

The Georgi-Glashow model was proposed as a theory of the weak interactions in 1972. It doesn't quite work, as you will see, but many key features of the electroweak theory are present. The Lagrangian for the Georgi-Glashow model is:

$$\mathcal{L}_{\text{GG}} = -\frac{1}{2}\text{Tr}(F_{\mu\nu}F^{\mu\nu}) + \text{Tr}\left[(D_\mu\phi)(D^\mu\phi)\right] + \mu^2\text{Tr}\phi^2 - \lambda(\text{Tr}\phi^2)^2 \tag{1}$$

The gauge group is $SU(2)$, and the scalar field $\phi = \phi^i T^i$ is a triplet (i.e. adjoint) whose components ϕ^i are real and

$$T^a = \frac{1}{2}\sigma^a \tag{2}$$

a) What values of ϕ minimize the potential?

b) By $SU(2)$ symmetry you can choose the vacuum to be $\langle\phi^1\rangle = \langle\phi^2\rangle = 0$, $\langle\phi^3\rangle = v$.

Show that this choice of vacuum "breaks" two of the generators of $SU(2)$

$$e^{i\alpha_{1,2}T^{1,2}}\langle\phi\rangle e^{-i\alpha_{1,2}T^{1,2}} \neq \langle\phi\rangle \tag{3}$$

and leaves one generator unbroken

$$e^{i\alpha_3 T^3}\langle\phi\rangle e^{-i\alpha_3 T^3} = \langle\phi\rangle \tag{4}$$

by using the infinitesimal versions of equations (3) and (4).

c) Show that you can make a gauge choice (known as unitary gauge) so that $\phi^1 = \phi^2 = 0$, $\phi^3 = v + h$.

d) In unitary gauge, show that two of the three gauge bosons get a mass $M_A^2 = g^2v^2$, and that one of the gauge bosons remains massless.

The Georgi-Glashow model can accommodate the massive $W^\pm$ bosons as well as the photon, but it has no Z boson. The Z boson is necessary to explain weak neutral current interactions, which were first observed in elastic electron-neutrino scattering.

Solution:

a)

$$\mathrm{Tr}\phi^2 = \phi^i\phi^j\mathrm{Tr}[T^iT^j] = \frac{1}{2}\sum(\phi^i)^2 \equiv \frac{1}{2}(\phi^i\phi_i)$$

Then the potential is

$$V(\phi) = -\frac{1}{2}\mu^2(\phi^i\phi_i) + \frac{1}{4}\lambda(\phi^i\phi_i)^2$$

The derivative is

$$V'(\phi)_i = -\mu^2\phi^i + \lambda(\phi^j\phi_j)\phi^i = 0$$

The solution to the above equation is

$$\phi^i\phi_i = \frac{\mu^2}{\lambda} \equiv v^2$$

b) The infinitesimal version is, not summing over the index

$$e^{i\alpha_iT^i}\langle\phi\rangle e^{-i\alpha_iT^i} = (1 + i\alpha_iT^i)\langle\phi\rangle(1 - i\alpha_iT^i) = \langle\phi\rangle + i\alpha_i[T^i, \langle\phi\rangle]$$

Thus we need to prove that

$$[T^3, \langle\phi\rangle] = 0,\ [T^{1,2}, \langle\phi\rangle] \neq 0$$

This is obvious because with the specific choice of vacuum, we have

$$\langle\phi\rangle = vT^3$$

c) We can parametrize the field as

$$\phi = e^{i\left(\frac{\xi_1}{v}T^1\right)+i\left(\frac{\xi_2}{v}T^2\right)}(v+h)T^3e^{-i\left(\frac{\xi_1}{v}T^1\right)-i\left(\frac{\xi_2}{v}T^2\right)}$$

Infinitesimally, this is the same as the original parametrization.

And we can perform gauge transformation so that

$$\phi = (v+h)T^3$$

d) In unitary gauge, we look at the kinetic term of ϕ, for covariant derivative we have

$$\begin{aligned} D_\mu\phi &= \partial_\mu(v+h)T^3 + igA^i_\mu(v+h)[T^i, T^3] \quad (5) \\ &= \partial_\mu(v+h)T^3 + gA^1_\mu(v+h)T^2 - gA^2_\mu(v+h)T^1 \quad (6) \end{aligned}$$

The term in the kinetic term

$$\mathrm{Tr}\left[(D_\mu\phi)(D^\mu\phi)\right]$$

that contributes to the gauge boson mass is

$$\mathrm{Tr}\left[(gA^1_\mu(v)T^2 - gA^2_\mu(v)T^1)(gA^{\mu,1}(v)T^2 - gA^{\mu,2}(v)T^1)\right] = \frac{1}{2}g^2v^2((A^1_\mu)^2 + (A^2_\mu)^2) \quad (7)$$

where we used that

$$\mathrm{Tr}[(T^{1,2})^2] = \frac{1}{2},\ T_1T_2 + T_2T_1 = 0$$

This shows that A_1 and A_2 have masses $m_A^2 = g^2v^2$, while the third gauge boson remains massless.

2 *The Higgs Sector*

This problem is essential for this course. The Lagrangian for the Higgs sector in the Standard Model is given by

$$\mathcal{L}_{\text{Higgs}} = (D_\mu H)^\dagger (D^\mu H) - V(H)$$

where H is an $SU(2)_L$ doublet of scalar fields and

$$V(H) = -\mu^2 H^\dagger H + \lambda (H^\dagger H)^2$$

is the $SU(2)_L \times U(1)_Y$ invariant potential.

Consider a constant value of H for which this potential is minimized. By $SU(2)$ symmetry we can always choose this value to be

$$\begin{pmatrix} 0 \\ \frac{v}{\sqrt{2}} \end{pmatrix} \tag{8}$$

where v is a constant. Recall that the $U(1)_Y$ charge of H is $\frac{1}{2}$.

a) Show that the generator of $SU(2)_L \times U(1)_Y$ that leaves the vacuum invariant is $T_3 + Y$.

Next consider small fluctuations of the Higgs field around this vacuum in the unitary gauge

$$H(x) = \begin{pmatrix} 0 \\ \frac{v+h(x)}{\sqrt{2}} \end{pmatrix}.$$

b) Show that the potential then takes on the form

$$V(h) = (\lambda v^2) h^2 + \lambda v h^3 + \frac{\lambda}{4} h^4.$$

c) Also show that the kinetic term for the Higgs boson in the Lagrangian becomes

$$\mathcal{L}_{\text{Kinetic}} = \frac{1}{2}(\partial_\mu h)(\partial^\mu h) + \frac{1}{8}\left(g' B_\mu - g X^3_\mu\right)\left(g' B^\mu - g X^{3\mu}\right)(v+h)^2$$
$$+ \frac{1}{8}\left(g X^1_\mu - i g X^2_\mu\right)\left(g X^{1\mu} + i g X^{2\mu}\right)(v+h)^2$$

which shows that some of the vector bosons gain mass terms.

d) If we perform a field redefinition, we want the kinetic terms for the vector bosons to remain canonical normalized. Show that correct mass-eigenstate for the neutral particle above is

$$Z_\mu = \cos\theta_W X^3_\mu - \sin\theta_W B_\mu$$

where

$$\cos\theta_W = \frac{g}{\sqrt{(g')^2 + g^2}}$$

is the Weinberg angle.

e) Verify that under a gauge transformation involving $U(1)_{EM}$ ($U(1)_{EM}$ is the unbroken subgroup of $SU(2)_L \times U(1)_Y$), X^1_μ and X^2_μ transform into each other, and thus

$$W^{\pm\mu} = \frac{1}{\sqrt{2}}\left(X^1_\mu \mp iX^2_\mu\right)$$

represent charged particles under $U(1)_{EM}$.

f) Verify that the masses of the three massive bosons are given by

$$m_W^2 = \frac{1}{4}g^2v^2 \quad \text{and} \quad m_Z^2 = \frac{1}{4}v^2((g')^2 + g^2)$$

g) What is the expression for the vector field A_μ (given in terms of g, g', X^a_μ and B_μ) that remains massless? What is it called?

h) In general, the covariant derivative of field with $SU(2)_L \times U(1)_Y$ charge that is an $SU(2)$ doublet is given by

$$D_\mu = \partial_\mu - igX^a_\mu T^a - ig'YB_\mu \tag{9}$$

where

$$T^a = \frac{1}{2}\sigma^a \tag{10}$$

Please rewrite this covariant derivative in terms of $W^\pm_\mu$, Z_μ, A_μ and obtain the following form:

$$D_\mu = \partial_\mu - i\frac{g}{\sqrt{2}}(W^+_\mu T^+ + W^-_\mu T^-) - i\frac{g}{\cos\theta_W}Z_\mu(T^3 - Q\sin^2\theta_W) - ieA_\mu Q \tag{11}$$

where we also define

$$T^\pm \equiv T^1 \pm iT^2 \tag{12}$$

and

$$Q \equiv T_3 + Y,\ e \equiv \frac{gg'}{\sqrt{g^2 + g'^2}}$$

Solution:

a)

$$\begin{aligned}(T_3 + Y)\begin{pmatrix} 0 \\ \frac{v}{\sqrt{2}} \end{pmatrix} &= \left(\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix} + \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}\right)\begin{pmatrix} 0 \\ \frac{v}{\sqrt{2}} \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ \frac{v}{\sqrt{2}} \end{pmatrix} \\ &= 0\end{aligned} \tag{13}$$

This implies that any transformation generated by $T_3 + Y$ leaves the vacuum in variant.

b) First we have

$$H^\dagger H = \frac{1}{2}(0, v+h)\begin{pmatrix} 0 \\ v+h \end{pmatrix} = \frac{1}{2}(v+h)^2 \tag{14}$$

To find the constant v, we minimized the function $V(x) = -\mu^2 x + \lambda x^2$ where $x \equiv H^\dagger H$. So we have $V'(x) = -\mu^2 + 2\lambda x = 0$ at minimum. So $x = \frac{\mu^2}{2\lambda}$, with our convention $x = \frac{v^2}{2}$. So we have $\mu^2 = \lambda v^2$.

$$\begin{aligned} V(h) &= -\mu^2 \frac{1}{2}(v+h)^2 + \lambda \frac{1}{4}(v+h)^4 \\ &= -\frac{\lambda v^2}{2}(v+h)^2 + \frac{\lambda}{4}(v+h)^4 \\ &= -\frac{\lambda v^2}{2}(v^2 + 2vh + h^2) + \frac{\lambda}{4}(v^4 + 4v^3 h + 6v^2 h^2 + 4vh^3 + h^4) \\ &= \frac{\lambda}{4}(h^4 + 4vh^3 + 4v^2 h^2 - v^4) \end{aligned} \tag{15}$$

c) The covariant derivative of Higgs is defined by

$$\begin{aligned} D_\mu H &= \partial_\mu H - ig\frac{\sigma^a}{2}X^a_\mu H - ig'\frac{1}{2}B_\mu H \\ &= \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ \partial_\mu h \end{pmatrix} - \frac{i}{2}\frac{1}{\sqrt{2}}\begin{pmatrix} gX^3_\mu + g'B_\mu & gX^1_\mu - igX^2_\mu \\ gX^1_\mu + igX^2_\mu & g'B_\mu - gX^3_\mu \end{pmatrix}\begin{pmatrix} 0 \\ v+h \end{pmatrix} \\ &= \frac{1}{\sqrt{2}}\begin{pmatrix} -\frac{i}{2}(v+h)(gX^1_\mu - igX^2_\mu) \\ \partial_\mu h - \frac{i}{2}(v+h)(g'B_\mu - gX^3_\mu) \end{pmatrix} \end{aligned} \tag{16}$$

The kinetic term is

$$(D_\mu H)^\dagger (D^\mu H) = \frac{1}{2}\left(\frac{i}{2}(v+h)(gX^1_\mu + igX^2_\mu), \partial_\mu h + \frac{i}{2}(v+h)(g'B_\mu - gX^3_\mu)\right)\begin{pmatrix} -\frac{i}{2}(v+h)(gX^{1\mu} - igX^{2\mu}) \\ \partial^\mu h - \frac{i}{2}(v+h)(g'B^\mu - gX^{3\mu}) \end{pmatrix} \tag{17}$$

$$= \frac{1}{8}(v+h)^2 g^2 (X^1_\mu + iX^2_\mu)(X^{1\mu} - iX^{2\mu}) + \frac{1}{2}\partial_\mu h \partial^\mu h + \frac{1}{8}(v+h)^2 (g'B_\mu - gX^3_\mu)(g'B^\mu - gX^{3\mu})$$

d) Consider the "mass term"

$$\frac{v^2}{8}\left(g'B_\mu - gX^3_\mu\right)\left(g'B^\mu - gX^{3\mu}\right)$$

Then the mass matrix in $\{B, X^3\}$ basis

$$M^2 = \frac{v^2}{4}\begin{pmatrix} g'^2 & gg' \\ gg' & g^2 \end{pmatrix}$$

The eigenvalues are 0 and $\frac{v^2}{4}(g'^2 + g^2)$. The corresponding eigenvectors are

$$\begin{pmatrix} \frac{g}{\sqrt{g^2+g'^2}} \\ -\frac{g'}{\sqrt{g^2+g'^2}} \end{pmatrix} \text{ and } \begin{pmatrix} \frac{g'}{\sqrt{g^2+g'^2}} \\ \frac{g}{\sqrt{g^2+g'^2}} \end{pmatrix}$$

The normalizations of these vectors are chosen so that the following matrix is orthogonal:

$$O = \begin{pmatrix} \frac{g}{\sqrt{g^2+g'^2}} & \frac{g'}{\sqrt{g^2+g'^2}} \\ -\frac{g'}{\sqrt{g^2+g'^2}} & \frac{g}{\sqrt{g^2+g'^2}} \end{pmatrix}$$

When we apply this matrix O to the vector:

$$\begin{pmatrix} B_\mu \\ X_\mu^3 \end{pmatrix}$$

We obtain the two fields:

$$\begin{aligned} A_\mu &= \cos\theta_W B_\mu + \sin\theta_W X_\mu^3 \\ Z_\mu &= -\sin\theta_W B_\mu + \cos\theta_W X_\mu^3 \end{aligned}$$

e) As we have seen previously the generator that corresponds to the remaining symmetry is $T_3 + Y = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. Thus the unitary transformation is given by

$$U = \begin{pmatrix} e^{i\theta(x)} & 0 \\ 0 & 1 \end{pmatrix}$$

Apply this to the gauge field(where we suppress the Lorentz vector indices and also only consider the X^1, X^2 terms, as the others donot mix with these two) we have

$$\begin{aligned} \sum_{i=1}^{2} X_\mu^i \cdot \frac{\sigma^i}{2} &\rightarrow U \sum_{i=1}^{2} X_\mu^i \cdot \frac{\sigma^i}{2} U^\dagger \qquad (18) \\ &= \begin{pmatrix} e^{i\theta(x)} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & X^1 - iX^2 \\ X^1 + iX^2 & 0 \end{pmatrix} \begin{pmatrix} e^{-i\theta(x)} & 0 \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 0 & (X^1 - iX^2)e^{i\theta} \\ (X^1 + iX^2)e^{-i\theta} & 0 \end{pmatrix} \end{aligned}$$

hence we can identify $W^\pm = \frac{1}{\sqrt{2}}(X^1 \mp iX^2)$ and we have $W^\pm \rightarrow e^{\pm i\theta} W^\pm$ under the $U(1)_{EM}$ transformation.

f) The mass terms are $\frac{1}{4}v^2 g^2 W_\mu^- W^{+\mu} + \frac{1}{8}v^2(g'^2 + g^2)Z^\mu Z_\mu$ with the new definition, and thus

$$m_W^2 = \frac{1}{4}g^2 v^2 \qquad \text{and} \qquad m_Z^2 = \frac{1}{4}v^2((g')^2 + g^2)$$

g) The other neutral particle is

$$A_\mu = \cos\theta_W B_\mu + \sin\theta_W X_\mu^3 \qquad (19)$$

It is orthogonal to Z^μ and remains massless, this is our photon.

h) As

$$T^{\pm} = \frac{1}{2}(\sigma^1 \pm i\sigma^2) \tag{20}$$

We have explicitly

$$T^{+} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$T^{-} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

The covariant derivative of field with $SU(2) \times U(1)$ charge is given by

$$\begin{aligned} D_\mu &= \partial_\mu - igX^a_\mu \frac{\sigma^a}{2} - ig'YB_\mu \\ &= \partial_\mu - i\frac{g}{2}\begin{pmatrix} 0 & X^1 - iX^2 \\ X^1 + iX^2 & 0 \end{pmatrix} - i(gT^3X^3_\mu + g'YB_\mu) \end{aligned} \tag{21}$$

As $W^{\pm} = \frac{1}{\sqrt{2}}(X^1 \mp iX^2)$, we immediately see that

$$D_\mu = \partial_\mu - i\frac{g}{\sqrt{2}}(W^+_\mu T^+ + W^-_\mu T^-) - i(gT^3X^3_\mu + g'YB_\mu) \tag{22}$$

Invert the following relation

$$\begin{aligned} A_\mu &= \cos\theta_W B_\mu + \sin\theta_W X^3_\mu \\ Z_\mu &= -\sin\theta_W B_\mu + \cos\theta_W X^3_\mu \end{aligned}$$

we have

$$X^3_\mu = \cos\theta_W Z_\mu + \sin\theta_W A_\mu \tag{23}$$

$$B_\mu = \cos\theta_W A_\mu - \sin\theta_W Z_\mu \tag{24}$$

So we have

$$\begin{aligned} gT^3X^3_\mu + g'YB_\mu &= gT^3(\cos\theta_W Z_\mu + \sin\theta_W A_\mu) + g'Y(\cos\theta_W A_\mu - \sin\theta_W Z_\mu) \\ &= Z_\mu(gT^3\cos\theta_W - g'Y\sin\theta_W) + A_\mu(gT^3\sin\theta_W + g'Y\cos\theta_W) \\ &= \frac{g}{\cos\theta_W}Z_\mu(T^3\cos^2\theta_W - Y\sin^2\theta_W) + eA_\mu Q \\ &= \frac{g}{\cos\theta_W}Z_\mu(T^3 - Q\sin^2\theta_W) + eA_\mu Q \end{aligned} \tag{25}$$

So the covariant derivative is

$$D_\mu = \partial_\mu - i\frac{g}{\sqrt{2}}(W^+_\mu T^+ + W^-_\mu T^-) - i\frac{g}{\cos\theta_W}Z_\mu(T^3 - Q\sin^2\theta_W) - ieA_\mu Q \tag{26}$$