CKM Matrix

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1 The CKM Matrix

Before introducing the Yukawa terms, the Standard Model fermion kinetic terms are

$$\begin{aligned}\mathcal{L}_{\text{kinetic}} &= \sum_{\psi=Q_L,L_L,e^R,u^R,d^R} \bar{\psi} i \not{D} \psi \\ &= i\bar{u}^i \not{\partial} u^i + i\bar{d}^i \not{\partial} d^i + i\bar{e}^i \not{\partial} e^i + i\bar{\nu}_L^i \not{\partial} \nu_L^i \\ &- \frac{g_s}{2} \bar{u}^i \lambda^A \not{G}^A u^i - \frac{g_s}{2} \bar{d}^i \lambda^A \not{G}^A d^i \\ &- \frac{g}{\sqrt{2}} \bar{u}_L^i \not{W}^+ d_L^i - \frac{g}{\sqrt{2}} \bar{\nu}_L^i \not{W}^+ e_L^i + h.c. \\ &- \sum_{\psi=u^i,d^i,e^i,\nu_L^i} \bar{\psi} g \sin\theta_w \not{A} Q_\psi \psi \\ &- \sum_{\psi=u^i,d^i,e^i,\nu_L^i} \bar{\psi} \frac{g}{\cos\theta_w} \not{Z} (T_{3L} P_L - Q_\psi \sin^2\theta_w) \psi\end{aligned} \tag{1}$$

where $h.c.$ denotes the Hermitian conjugate. Let us consider the most general Yukawa Lagrangian below:

$$\mathcal{L}_{\text{Yukawa}} = -Y_{ij}^{(u)} \bar{u}_R^i (Q_L^j)^T \epsilon H - Y_{ij}^{(d)} H^\dagger \bar{d}_R^i Q_L^j - Y_{ij}^{(e)} H^\dagger \bar{e}_R^i L_L^j + h.c.$$

When the Higgs field H takes on a vacuum expectation value (vev), in unitary gauge, we have

$$H = \begin{pmatrix} 0 \\ \frac{v+h}{\sqrt{2}} \end{pmatrix}$$

the Yukawa Lagrangian will take the form of

$$\mathcal{L}_{\text{Yukawa}} = -Y_{ij}^{(u)} \frac{v}{\sqrt{2}} \bar{u}_R^i u_L^j \left(1 + \frac{h}{v}\right) - Y_{ij}^{(d)} \frac{v}{\sqrt{2}} \bar{d}_R^i d_L^j \left(1 + \frac{h}{v}\right) - Y_{ij}^{(e)} \frac{v}{\sqrt{2}} \bar{e}_R^i e_L^j \left(1 + \frac{h}{v}\right)$$

and will give rise to the masses for all of the fermions except for the neutrinos. Keeping this in mind we want to diagonalize the Yukawa matrices $Y^{(u),(d),(e)}$. In other words we want to write the Yukawa matrices as:

$$M^{(u)} \equiv \frac{Y^{(u)}}{\sqrt{2}} = U_u m^{(u)} V_u^\dagger$$
$$M^{(d)} \equiv \frac{Y^{(d)}}{\sqrt{2}} = U_d m^{(d)} V_d^\dagger$$
$$M^{(e)} \equiv \frac{Y^{(e)}}{\sqrt{2}} = U_e m^{(e)} V_e^\dagger$$

where m are the diagonal mass matrices for the quarks and leptons. U and V are unitary matrices.

We will attempt to diagonalize the Yukawa matrices:

a) We are going to start with the leptons. Show that by using the $U(3)_{e_L} \times U(3)_{e_R}$ global symmetry for e^i_L and e^i_R, we can diagonalize the last term in $\mathcal{L}_{\text{Yukawa}}$. Show that by doing so $U(3)_{e_L} \times U(3)_{e_R}$ breaks down to $U(1)^3$. (One $U(1)$ for each generation.) The conserved charge associated with this unbroken symmetry is called the *lepton family number*.

b) We can now use the $U(3)$ rotations associated with d^i_R and d^i_L to diagonalize the Yukawa terms involving the d^i_R terms. (The second term in $\mathcal{L}_{\text{Yukawa}}$). Write out the $U(3)$ rotations.

c) Show that it is not possible to diagonalize simultaneously the first term in the Yukawa Lagrangian and the kinetic terms using symmetry arguments. Show that we can still diagonalize the first term in the Yukawa Lagrangian, but one of the kinetic terms will pick up a matrix (known as the CKM (Cabibbo-Kobayashi-Maskawa) matrix) that mixes the different generations of quarks. Show that CKM matrix is unitary.

d) How many parameters in the CKM matrix are physically relevant? To answer this question, it is instructional to consider two generations first. A unitary 2×2 matrix can be parameterized with one angle and three phases:

$$V = \begin{pmatrix} \cos\theta_c e^{i\alpha} & \sin\theta_c e^{i\beta} \\ -\sin\theta_c e^{i(\alpha+\gamma)} & \cos\theta_c e^{i(\beta+\gamma)} \end{pmatrix} \tag{2}$$

Which of these parameters can be absorbed by quark field redefinition? How many physically significant parameters are we left with? How many are angles and how many are phases?

e) Now consider three generations. How many physically significant parameters do we have now? How many are angles and how many are phases? Comment on the difference that arises from adding a generation other than the change in the number of parameters. Why is this significant? (Hint: Time reversal is an anti-unitary operator).

f) In the lepton sector the Yukawa terms break the $U(3)_{L_L} \times U(3)_{e_R}$ global symmetry to a $U(1)$ symmetry within each generation. Show that, in contrast, in the quark sector the CKM matrix breaks the corresponding global $U(3)^3$ to a single $U(1)$ symmetry. The conserved charge corresponds to this symmetry is the quark number or one-third of the baryon number.

g) Under what circumstances would the leptonic sector have a mixing matrix analogous to the CKM matrix?

2 Gauge Invariance, Lorentz Invariance, and Renormalizability

For each of the following terms, determine if the term is Lorentz invariant. If it is Lorentz invariant, determine if the term is gauge invariant or can be made gauge invariant by the insertion of one or more ϵ's. If it is not gauge invariant, state which gauge symmetry or symmetries the term violates. If the term is both Lorentz invariant and gauge invariant, determine if it is renormalizable.

a) $\bar{Q}_L Q_L$

b) $\bar{L}_L L_L$

c) $\bar{e}_R e_R$

d) $\bar{e}_R \not{D} e_R$

e) $H^\dagger H$

f) $H L_L$

g) $\bar{e}_R H^\dagger L_L$

h) $\bar{u}_R H^\dagger Q_L$

i) $\bar{e}_R H^\dagger Q_L$

j) $Q_L^T \gamma^\mu Q_L Q_L^T \gamma_\mu L_L$

k) $\bar{u}_L \bar{d}_L \bar{Q}_L H$

l) $L_L^T \gamma^2 \gamma^0 L_L H H$