CKM Matrix

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1 The CKM Matrix

Before introducing the Yukawa terms, the Standard Model fermion kinetic terms are

$$\begin{aligned}\mathcal{L}_{\text{kinetic}} &= \sum_{\psi=Q_L,L_L,e^R,u^R,d^R} \bar{\psi} i \not{D} \psi \\ &= i\bar{u}^i \not{\partial} u^i + i\bar{d}^i \not{\partial} d^i + i\bar{e}^i \not{\partial} e^i + i\bar{\nu}^i_L \not{\partial} \nu^i_L \\ &- \frac{g_s}{2}\bar{u}^i \lambda^A \not{G}^A u^i - \frac{g_s}{2}\bar{d}^i \lambda^A \not{G}^A d^i \\ &- \frac{g}{\sqrt{2}}\bar{u}^i_L \not{W}^+ d^i_L - \frac{g}{\sqrt{2}}\bar{\nu}^i_L \not{W}^+ e^i_L + h.c. \\ &- \sum_{\psi=u^i,d^i,e^i,\nu^i_L} \bar{\psi} g \sin\theta_w \not{A} Q_\psi \psi \\ &- \sum_{\psi=u^i,d^i,e^i,\nu^i_L} \bar{\psi} \frac{g}{\cos\theta_w} \not{Z}(T_{3L}P_L - Q_\psi \sin^2\theta_w)\psi\end{aligned} \tag{1}$$

where $h.c.$ denotes the Hermitian conjugate. Let us consider the most general Yukawa Lagrangian below:

$$\mathcal{L}_{\text{Yukawa}} = -Y^{(u)}_{ij}\bar{u}^i_R(Q^j_L)^T\epsilon H - Y^{(d)}_{ij}H^\dagger\bar{d}^i_R Q^j_L - Y^{(e)}_{ij}H^\dagger\bar{e}^i_R L^j_L + h.c.$$

When the Higgs field H takes on a vacuum expectation value (vev), in unitary gauge, we have

$$H = \begin{pmatrix} 0 \\ \frac{v+h}{\sqrt{2}} \end{pmatrix}$$

the Yukawa Lagrangian will take the form of

$$\mathcal{L}_{\text{Yukawa}} = -Y^{(u)}_{ij}\frac{v}{\sqrt{2}}\bar{u}^i_R u^j_L\left(1+\frac{h}{v}\right) - Y^{(d)}_{ij}\frac{v}{\sqrt{2}}\bar{d}^i_R d^j_L\left(1+\frac{h}{v}\right) - Y^{(e)}_{ij}\frac{v}{\sqrt{2}}\bar{e}^i_R u^j_L\left(1+\frac{h}{v}\right)$$

and will give rise to the masses for all of the fermions except for the neutrinos. Keeping this in mind we want to diagonalize the Yukawa matrices $Y^{(u),(d),(e)}$[1]. In other words we want to write the

[1] Why did Mouse win a Nobel prize for explaining "why chicken tastes like everything" to Neo?

Yukawa matrices as:

$$M^{(u)} \equiv \frac{Y^{(u)}}{\sqrt{2}} = U_u m^{(u)} V_u^\dagger$$
$$M^{(d)} \equiv \frac{Y^{(d)}}{\sqrt{2}} = U_d m^{(d)} V_d^\dagger$$
$$M^{(e)} \equiv \frac{Y^{(e)}}{\sqrt{2}} = U_e m^{(e)} V_e^\dagger$$

where m are the diagonal mass matrices for the quarks and leptons. U and V are unitary matrices.

We will attempt to diagonalize the Yukawa matrices:

a) We are going to start with the leptons. Show that by using the $U(3)_{e_L} \times U(3)_{e_R}$ global symmetry for e_L^i and e_R^i, we can diagonalize the last term in $\mathcal{L}_{\text{Yukawa}}$. Show that by doing so $U(3)_{e_L} \times U(3)_{e_R}$ breaks down to $U(1)^3$. (One $U(1)$ for each generation.) The conserved charge associated with this unbroken symmetry is called the *lepton family number*.

b) We can now use the $U(3)$ rotations associated with d_R^i and d_L^i to diagonalize the Yukawa terms involving the d_R^i terms. (The second term in $\mathcal{L}_{\text{Yukawa}}$). Write out the $U(3)$ rotations.

c) Show that it is not possible to diagonalize simultaneously the first term in the Yukawa Lagrangian and the kinetic terms using symmetry arguments. Show that we can still diagonalize the first term in the Yukawa Lagrangian, but one of the kinetic terms will pick up a matrix (known as the CKM (Cabibbo-Kobayashi-Maskawa) matrix) that mixes the different generations of quarks. Show that CKM matrix is unitary.

d) How many parameters in the CKM matrix are physically relevant? To answer this question, it is instructional to consider two generations first. A unitary 2×2 matrix can be parameterized with one angle and three phases:

$$V = \begin{pmatrix} \cos\theta_c e^{i\alpha} & \sin\theta_c e^{i\beta} \\ -\sin\theta_c e^{i(\alpha+\gamma)} & \cos\theta_c e^{i(\beta+\gamma)} \end{pmatrix} \tag{2}$$

Which of these parameters can be absorbed by quark field redefinition? How many physically significant parameters are we left with? How many are angles and how many are phases?

e) Now consider three generations. How many physically significant parameters do we have now? How many are angles and how many are phases? Comment on the difference that arises from adding a generation other than the change in the number of parameters. Why is this significant? (Hint: Time reversal is an anti-unitary operator).

f) In the lepton sector the Yukawa terms break the $U(3)_{L_L} \times U(3)_{e_R}$ global symmetry to a $U(1)$ symmetry within each generation. Show that, in contrast, in the quark sector the CKM matrix [2] breaks the corresponding global $U(3)^3$ to a single $U(1)$ symmetry. The conserved charge corresponds to this symmetry is the quark number or one-third of the baryon number.

g) Under what circumstances would the leptonic sector have a mixing matrix analogous to the CKM matrix?

[2] He discovered that the Matrix mixes flavors.

Solution:

a) We can use the SVD process to diagonalize $M^{(e)} = U_e m^{(e)} V_e^\dagger$, with this change the corresponding Yukawa term looks like

$$(U_e m^{(e)} V_e^\dagger)_{ij} \bar{e}_R^i e_L^j \tag{3}$$

Thus we can redefine our new field as

$$e_L^{\prime,j} \equiv V_{e,jk}^\dagger e_L^k, \quad e_R^{\prime,i} \equiv (U_e^\dagger)_{il} e_R^l \tag{4}$$

In terms of the new fields, the Yukawa coupling will look like

$$\sum_i m_{ii}^{(e)} \bar{e}_R^{\prime,i} e_L^{\prime i} \tag{5}$$

We can drop the primes now, but we cannot do arbitrary $U(3)$ transformations any more as that will spoil the Yukawa term. Note that this is still invariant under the following transformation for each generation,

$$e_L^i \to e^{i\alpha_i} e_L^i, \quad e_R^i \to e^{i\alpha_i} e_R^i \tag{6}$$

Thus we find a remaining global symmetry of $U(1)^3$. The corresponding conserved charge is the lepton family number.

b) We can proceed the exactly same way, by redefining d_R^i and d_L^i as follows.

$$d_L^{\prime,j} \equiv V_{d,jk}^\dagger d_L^k, \quad d_R^{\prime,i} \equiv (U_d^\dagger)_{il} d_R^l \tag{7}$$

and we will achieve the following diagonalized Yukawa term,

$$\sum_i M_{ii}^{(d)} \bar{d}_R^{\prime,i} d_L^{\prime i} \tag{8}$$

c) We want to proceed using the same strategy as before by redefining the fields,

$$u_L^{\prime,j} \equiv V_{u,jk}^\dagger u_L^k, \quad u_R^{\prime,i} \equiv (U_u^\dagger)_{il} u_R^l \tag{9}$$

but we cannot do this, because we already used the $U(3)$ symmetry of Q_Ls and there is no reason to believe U_d and U_u coincide. Recall that the Lagrangian for the kinetic terms for fermions can also be written in the following form where the global symmetry is evidently $U(3)^5$,

$$\mathcal{L}_{\text{fermi}} = \sum_{\psi = Q_L, L_L, e^R, u^R, d^R} \bar{\psi} i \not{D} \psi$$

We can still proceed we and define u fields as above, but in terms of the new fields, the term in the kinetic term that couples to W boson will not be diagonal in the flavor basis. Originally, they are written as follows

$$-\frac{g}{\sqrt{2}} \bar{u}_L^i \not{W}^+ d_L^i - \frac{g}{\sqrt{2}} \bar{\nu}_L^i \not{W}^+ e_L^i + h.c. \tag{10}$$

Using the transformation just proposed, in terms of the new fields, we have

$$\bar{u}_L^{\prime,j} V_{u,ji}^\dagger \not{W}^+ V_{d,ki} d_L^{\prime,k} \tag{11}$$

and its hermitian conjugate term. The CKM matrix is defined to be

$$V_{\mathrm{CKM}} = V_u^\dagger V_d \tag{12}$$

So that we have W boson interaction as

$$\bar{u}_L^{\prime,i} V_{\mathrm{CKM},ij} \not{W}^+ d_L^{\prime,j} \tag{13}$$

and its hermitian conjugate. This is unitary as $V_{\mathrm{CKM}} V_{\mathrm{CKM}}^\dagger = 1$ from definition of U_d and U_u's. It has 9 real parameters.

d) For the cases with 2 generations, the matrix has 4 real parameter. Redefine the four quark field can absorb three phases and the angle will be physically important parameter that remains. The overall phase is a symmetry (as explored later on).

e) We can write a general 3×3 unitary matrix as:

$$V_{CKM} = V_1 V_2 V_3$$

where V_1 and V_3 are diagonal matrices of six phases and V_2 is a three orthogonal matrix. We can rephase our fermionic fields in such a way as to absorb five of these six phases. The Lagrangian still has an overall phase symmetry (see below) which cannot be used to absorb one of the phases, and so there will still be a phase left over in the CKM matrix. The V_2 part contributes three real angles. so there will be 4 independent physical parameters in CKM matrices. The difference from the two generation scenario is that we have a phase that cannot be absorbed. The phase will make the W boson interaction Lagrangian not invariant under time reversal transformation. As CPT is always invariant. Time reversal violation is also known as CP violation.

f) Since now the Yukawa term contains CKM matrix which mixes the generations. We cannot have $U(1)$ symmetries for each generation as for leptons. But we can still have the proposed overall U(1) symmetry because

$$Q_l (V_{\mathrm{CKM}})_{lk} M_{kj}^{(u)} u_{j,R}^C \tag{14}$$

is invariant under

$$\begin{aligned} Q_i &\to e^{i\alpha} Q_i \\ \bar{u}_i &\to e^{-i\alpha} \bar{u}_i \\ \bar{d}_i &\to e^{-i\alpha} \bar{d}_i. \end{aligned}$$

Notice that for leptons we can have three α_is, now we only have one α shared across three families.

g) If we introduce right handed neutrinos, we will have another term in the Yukawa Lagrangian and face the problem we have before for quarks. This is known in literature as PMNS matrix.

2 Gauge Invariance, Lorentz Invariance, and Renormalizability

For each of the following terms, determine if the term is Lorentz invariant. If it is Lorentz invariant, determine if the term is gauge invariant or can be made gauge invariant by the insertion of one or more ϵ's. If it is not gauge invariant, state which gauge symmetry or symmetries the term violates. If the term is both Lorentz invariant and gauge invariant, determine if it is renormalizable.

a) $\bar{Q}_L Q_L$

b) $\bar{L}_L L_L$

c) $\bar{e}_R e_R$

d) $\bar{e}_R \not{D} e_R$

e) $H^\dagger H$

f) $H L_L$

g) $\bar{e}_R H^\dagger L_L$

h) $\bar{u}_R H^\dagger Q_L$

i) $\bar{e}_R H^\dagger Q_L$

j) $Q_L^T \gamma^\mu Q_L Q_L^T \gamma_\mu L_L$

k) $\bar{u}_L \bar{d}_L \bar{Q}_L H$

l) $L_L^T \gamma^2 \gamma^0 L_L H H$

Solution:

a) $\bar{Q}Q$ This is zero as

$$Q_L = \frac{1+\gamma^5}{2} Q$$

$$\bar{Q}_L Q_L = Q^\dagger \frac{1+\gamma^5}{2} \gamma_0 \frac{1+\gamma^5}{2} Q = Q^\dagger \gamma_0 \frac{1-\gamma^5}{2} \frac{1+\gamma^5}{2} Q = 0$$

b) $\bar{L}_L L_L$ For the same reason as part a) this is also zero

c) $\bar{e}_R e_R$ For the same reason as part a) this is also zero

d) $\bar{e}_R \not{D} e_R$ This exists as part of the kinetic term in the SM Lagrangian, it is Lorentz invariant, gauge invariant and renormalizable.

e) $H^\dagger H$ This exists as part of the potential term in the SM Lagrangian, it is Lorentz invariant, gauge invariant and renormalizable.

f) $H L_L$ This is not Lorentz invariant, as fermions need to show up in pairs.

g) $\bar{e}_R H^\dagger L_L$ This is the Yukawa term in the SM Lagrangian to generate electron mass, it is Lorentz invariant, gauge invariant and renormalizable.

h) $\bar{u}_R H^\dagger Q_L$ It is Lorentz invariant, but it is not gauge invariant, it breaks U(1) gauge symmetry, the total hyper charge is $-\frac{2}{3} - \frac{1}{2} + \frac{1}{6} \neq 0$.

i) $\bar{e}_R H^\dagger Q_L$ This is Lorentz invariant, but it is not gauge invariant, it breaks SU(3) gauge symmetry as only quark carries colors here.

j) $Q_L^T \gamma^\mu Q_L Q_L^T \gamma_\mu L_L$ This is Lorentz invariant, but it is not gauge invariant, it breaks both SU(3) and SU(2) gauge symmetry, to make it gauge invariant, we need two ϵ tensor to make it $SU(2)$ invariant and one more ϵ tensor to make it $SU(3)$ invariant. It is not renormalizable as it has dimension six.

k) $\bar{u}_L \bar{d}_L \bar{Q}_L H$ This is not Lorentz invariant, as it has odd number of fermions.

l) $L_L^T \gamma^2 \gamma^0 L_L H H$ This is Lorentz invariant, but it is not gauge invariant, as it breaks the SU(2) gauge symmetry, to make it gauge invariant, we need two ϵ tensor to make it $SU(2)$ invariant. It is not reenormalizable as it has dimension five.