Deep Inelastic Scattering

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1 Charged Current Deep Inelastic Scattering

a) To understand charged current deep inelastic scattering, we first need to find the subprocess differential cross sections. By neglecting all fermion masses, show that

$$\frac{d\hat{\sigma}}{dy}(e^- q \to \nu q') = \frac{1}{2}\frac{d\hat{\sigma}}{dy}(\nu_e q' \to e^- q) = \frac{G_F^2 \hat{s}}{2\pi}|V_{qq'}|^2 \left(\frac{M_W^2}{M_W^2 + Q^2}\right)^2 \tag{1}$$

Quantities associated with the subprocess conventionally wear hats, e.g. the Mandelstam invariants for the subprocess are $\hat{s}, \hat{t}, \hat{u}$. The Mandelstam variable $\hat{t}$ has an alias: $Q^2 = -\hat{t}$. The dimensionless variable y is given by $y = -\hat{t}/\hat{s}$. The Fermi coupling is related to the $SU(2)_L$ coupling by $\frac{G_F}{\sqrt{2}} = \frac{g^2}{8M_W^2}$. Where does the factor of $\frac{1}{2}$ come from? If we assume the scales at which the scatterings happen are much lower than the mass the of the W boson, again we see that because of factorization, the fundamental scattering process is elastic.

b) Show that the parton differential cross sections with anti-neutrinos are given by

$$\frac{1}{2}\frac{d\hat{\sigma}}{dy}(\bar{\nu}_\mu q \to \mu^+ q') = \frac{d\hat{\sigma}}{dy}(\mu^+ q' \to \bar{\nu}_\mu q) = \frac{G_F^2 \hat{s}}{2\pi}|V_{qq'}|^2 \left(\frac{M_W^2}{M_W^2 + Q^2}\right)^2 (1-y)^2 \tag{2}$$

The difference between the neutrino and anti-neutrino cross sections shows that the weak interaction is chiral. For instance in class we derive the double differential cross section for the process of $e^- p \to e^- X$, the result would not change for $e^+ p \to e^+ X$. For more discussion about how helicity of the particles affect which Mandelstam variables show up in the cross section, we refer interested reader to section 5.2 of Peskin and Schroeder. Note that the result from the last part does not depend on y. The dependence on y for both neutrino and anti-neutrino scattering has been experimentally tested.

c) Most of the momentum in protons and neutrons is carried by the valence quarks (u and d) and by gluons. The remainder of the momentum is carried by the sea quarks (s, c, b, $\bar{u}$, $\bar{d}$,$\bar{s}$, $\bar{c}$, and $\bar{b}$). By neglecting the sea quarks (i.e. in the valence dominance approximation) and using the fact that the CKM matrix is close to the identity matrix show that

$$\begin{aligned} \frac{d\sigma(\nu_\mu p \to \mu^- X)}{dx} &= \frac{G_F^2 s}{\pi} x f_d(x) \\ \frac{d\sigma(\nu_\mu n \to \mu^- X)}{dx} &= \frac{G_F^2 s}{\pi} x f_u(x) \end{aligned} \tag{3}$$

where $f_u(x)$ and $f_d(x)$ are the up and down quark parton distribution functions in the proton, and $x = \frac{Q^2}{2P\cdot q}$, P is the nucleon momentum, and q is the W boson momentum. The PDF's of hadrons reflect the symmetries that relate different hadrons. Since the neutron and proton are related under the exchange of the u and d quarks, $f_u(x) \approx f_d^n(x)$ and $f_d(x) \approx f_u^n(x)$ where $f_u^n(x)$ and $f_d^n(x)$ are the parton distribution functions for the up and down quarks respectively in the neutron.

d) Use your results and the result from lecture

$$\frac{d^2\sigma(e^- p \to e^- X)}{dxdy} = \sum_q x f_q(x) \frac{2\pi\alpha^2 s Q_q^2}{Q^4}\left(y^2 + 2(1-y)\right) \tag{4}$$

and the following definition of the form factors

$$F_2^{\nu D}(x) = \left(\frac{2\pi}{G_F^2 s}\right)\left[\frac{d\sigma(\nu_\mu p \to \mu X)}{dx} + \frac{d\sigma(\nu_\mu n \to \mu X)}{dx}\right] \tag{5}$$

and

$$F_2^{eD}(x) = \left(\frac{3Q^4}{8\pi\alpha^2 s}\right)\left[\frac{d\sigma(e^- p \to e^- X)}{dx} + \frac{d\sigma(e^- n \to e^- X)}{dx}\right] \tag{6}$$

to show that in the valence dominance approximation

$$F_2^{\nu D}(x) = K F_2^{eD}(x) \tag{7}$$

where K is a constant, find K assuming the charge for up quark is Q_u and the charge for the down quark is Q_d.

e) Experimentally $K \approx \frac{18}{5}$. What does this tell you about the quark charges?

2 Useful Formulae

$$\gamma^\mu\gamma_\mu = 4 \tag{8}$$
$$\gamma^\mu\gamma^\nu\gamma_\mu = -2\gamma^\nu \tag{9}$$
$$\gamma^\mu\gamma^\nu\gamma^\lambda\gamma_\mu = 4g^{\nu\lambda} \tag{10}$$
$$\gamma^\mu\gamma^\nu\gamma^\lambda\gamma^\sigma\gamma_\mu = -2\gamma^\sigma\gamma^\lambda\gamma^\nu \tag{11}$$
$$\mathrm{tr}\,\gamma^\mu\gamma^\nu\gamma^\lambda = 0 \tag{12}$$
$$\text{the same for arbitrary odd number of } \gamma^{\mu\prime}\text{s} \tag{13}$$
$$\mathrm{tr}\,\gamma^\mu\gamma^\nu = 4g^{\mu\nu} \tag{14}$$
$$\mathrm{tr}\,\gamma^\mu\gamma^\nu\gamma^\lambda\gamma^\rho = 4\left(g^{\mu\nu}g^{\lambda\rho} + g^{\mu\rho}g^{\nu\lambda} - g^{\mu\lambda}g^{\nu\rho}\right) \tag{15}$$
$$\mathrm{tr}\gamma^\mu\gamma^\nu\gamma^5 = 0 \tag{16}$$
$$\mathrm{tr}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma^5 = -4i\epsilon^{\mu\nu\rho\sigma} \tag{17}$$

Solution:

a) First we calculate the matrix element for the process, which is the same for both cases,

$$\begin{aligned} i\mathcal{M} &= \left(\frac{-ig}{\sqrt{2}}V_{qq'}\right)\left(\frac{-ig}{\sqrt{2}}\right)(\bar{u}_{q'}\gamma^\mu P_L u_q)\frac{\eta_{\mu\nu} - \frac{k_\mu k_\nu}{M_W^2}}{M_W^2 - q^2}(\bar{u}_{\nu_e}\gamma^\nu P_L u_e) \qquad (18)\\ &= \left(\frac{-ig}{\sqrt{2}}V_{qq'}\right)\left(\frac{-ig}{\sqrt{2}}\right)\frac{1}{M_W^2 - q^2}\left((\bar{u}_{q'}\gamma^\mu P_L u_q)(\bar{u}_{\nu_e}\gamma_\mu P_L u_e) - \frac{1}{M_W^2}(\bar{u}_{q'}\not{k}P_L u_q)(\bar{u}_{\nu_e}\not{k}P_L u_e)\right)\\ &= \left(\frac{-ig}{\sqrt{2}}V_{qq'}\right)\left(\frac{-ig}{\sqrt{2}}\right)\frac{1}{M_W^2 - q^2}\left((\bar{u}_{q'}\gamma^\mu P_L u_q)(\bar{u}_{\nu_e}\gamma_\mu P_L u_e)\right) \end{aligned}$$

where we dropped the last term as we assume massless fermions.

Let us sum some spins,

$$(\bar{u}_{q'}\gamma^\mu P_L u_q)(\bar{u}_q\gamma^\nu P_L u_{q'}) = \mathrm{Tr}[\not{q}\gamma^\mu(1-\gamma^5)\not{q}'\gamma^\nu(1-\gamma^5)] \tag{19}$$

The matrix element square is given by

$$|\mathcal{M}|^2 = \left(\frac{g^2}{8}\right)^2|V_{qq'}|^2\left(\frac{1}{M_W^2 + q^2}\right)^2\mathrm{Tr}[\not{q}\gamma^\mu(1-\gamma^5)\not{q}'\gamma^\nu(1-\gamma^5)]\mathrm{Tr}[\not{q}_{\nu_e}\gamma_\mu(1-\gamma^5)\not{q}_e\gamma_\nu(1-\gamma^5)]$$

We can calculate the trace,

$$\begin{aligned} \mathrm{Tr}[\not{q}\gamma^\mu(1-\gamma^5)\not{q}'\gamma^\nu(1-\gamma^5)] &= 2\mathrm{Tr}[\not{q}\gamma^\mu\not{q}'\gamma^\nu] - \mathrm{Tr}[\not{q}\gamma^\mu\not{q}'\gamma^\nu\gamma^5] - \mathrm{Tr}[\not{q}\gamma^\mu\gamma^5\not{q}'\gamma^\nu] \qquad (20)\\ &= 2\times 4(q^\mu q'^\nu + q^\nu q'^\mu - g^{\mu\nu}q\cdot q') + 4i\epsilon^{\rho\mu\sigma\nu}q_\rho q'_\sigma + 4i\epsilon^{\sigma\nu\rho\mu}q_\rho q'_\sigma\\ &= 2\times 4(q^\mu q'^\nu + q^\nu q'^\mu - g^{\mu\nu}q\cdot q' + i\epsilon^{\rho\mu\sigma\nu}q_\rho q'_\sigma) \end{aligned}$$

Similarly,

$$\mathrm{Tr}[\not{q}_{\nu_e}\gamma_\mu(1-\gamma^5)\not{q}_e\gamma_\nu(1-\gamma^5)] = 2\times 4(q_{\nu_e,\mu}q_{e,\nu} + q_{\nu_e,\nu}q_{e,\mu} - g_{\mu\nu}q_{\nu_e}\cdot q_e + i\epsilon_{\rho\mu\sigma\nu}q_{\nu_e}^\rho q_e^\sigma) \tag{21}$$

Thus we have

$$
\begin{aligned}
|\mathcal{M}|^2 &= \left(\frac{g^2}{8}\right)^2 |V_{qq'}|^2 \left(\frac{1}{M_W^2+q^2}\right)^2 \\
&\times 64(2(q\cdot q_{\nu_e})(q'\cdot q_e) + 2(q'\cdot q_{\nu_e})(q\cdot q_e) - 2((q\cdot q_{\nu_e})(q'\cdot q_e) - (q'\cdot q_{\nu_e})(q\cdot q_e))) \\
&= \left(\frac{g^2}{8}\right)^2 |V_{qq'}|^2 \left(\frac{1}{M_W^2+q^2}\right)^2 64\times 4(q'\cdot q_{\nu_e})(q\cdot q_e) \\
&= g^4|V_{qq'}|^2 \left(\frac{1}{M_W^2+q^2}\right)^2 \hat{s}^2
\end{aligned} \tag{22}
$$

Using

$$\left(\frac{d\sigma}{-d\cos\theta_{CM}d\phi}\right)_{CM} = \frac{|\mathcal{M}|^2}{64\pi^2E_{cm}^2}$$

and

$$\hat{t} = -\hat{s}\frac{1-\cos\theta_{CM}}{2}$$

so

$$dy = -\frac{1}{2}d\cos\theta_{CM}$$

We have

$$
\begin{aligned}
\frac{d\hat{\sigma}}{dy} &= \frac{1}{4}\frac{|\mathcal{M}|^2}{64\pi^2E_{cm}^2}\times 2\pi\times 2 \\
&= \frac{1}{4}g^4|V_{qq'}|^2\left(\frac{1}{M_W^2+q^2}\right)^2\hat{s}^2\frac{1}{16\pi\hat{s}} \\
&= \frac{g^4}{64\pi}|V_{qq'}|^2\left(\frac{1}{M_W^2+q^2}\right)^2\hat{s} \\
&= \frac{G_F^2\hat{s}}{2\pi}|V_{qq'}|^2\left(\frac{M_W^2}{M_W^2+Q^2}\right)^2
\end{aligned} \tag{23}
$$

First line we take care of the difference between $d\Omega$ and dy and last line we used definition of the Fermi constant. This is the differential cross section of electron scattered off quark to neutrino, so we average over both spins of quarks and electrons. For the process of scattering neutrinos off quarks, we only need to average of the quark, because neutrino is massless, and only left-handed neutrino exists in Standard model.

b) We will still calculate the process of scattering an anti-lepton of a quark, as this is related to the anti-neutrino scattering by a factor of 2 and it is most similar to what we have calculated, actually all the steps in last part fall through, except, now the fermion lepton line become anti-particles and the direction is reversed, so that relevant part of the Matrix element is,

$$i\mathcal{M} = \ldots(\bar{v}_\mu\gamma_\mu P_L v_{\nu_\mu})$$

That is, compare to the previous case u becomes v and the order of μ and ν_μ are changed. Thus, the only thing we need to change in the $|\mathcal{M}|^2$ is to switch the neutrino and electron momentum, that will change $\hat{s}^2$ to $\hat{u}^2$,

$$
\begin{aligned}
|\mathcal{M}|^2 &= \left(\frac{g^2}{8}\right)^2 |V_{qq'}|^2 \left(\frac{1}{M_W^2+q^2}\right)^2 64 \times 4(q' \cdot q_\mu)(q \cdot q_{\nu_\mu}) \\
&= g^4 |V_{qq'}|^2 \left(\frac{1}{M_W^2+q^2}\right)^2 \hat{u}^2
\end{aligned}
\tag{24}
$$

And it follows that the differential cross section will be

$$
\begin{aligned}
\frac{d\hat{\sigma}}{dy} &= \frac{G_F^2 \hat{s}}{2\pi} |V_{qq'}|^2 \left(\frac{M_W^2}{M_W^2+Q^2}\right)^2 \frac{\hat{u}^2}{\hat{s}^2} \\
&= \frac{G_F^2 \hat{s}}{2\pi} |V_{qq'}|^2 \left(\frac{M_W^2}{M_W^2+Q^2}\right)^2 (1-y)^2
\end{aligned}
\tag{25}
$$

c) In the valence dominance approximation in the process of neutrinos scattering of quarks, the only subprocess is $\nu_\mu d \to \mu u$. As $V_{ud} \approx 1$ and M_W^2 dominates, we only need to multiply by $f_d(x)dx$, which for neutron's case $f_{n,d}(x) \approx f_{p,u}(x)$.

d) For the scattering through photon case, with valence dominance approximation, we have

$$
\frac{d^2\sigma(ep \to eX)}{dxdy} = x\left(Q_u^2 f_u(x) + (1-2Q_u)^2 f_d(x)\right) \frac{2\pi\alpha^2 s}{Q^4} \left(y^2 + 2(1-y)\right)
$$

Perform the integral over y we have,

$$
\begin{aligned}
\frac{d^2\sigma(ep \to eX)}{dx} &= x\left(Q_u^2 f_u(x) + (1-2Q_u)^2 f_d(x)\right) \frac{2\pi\alpha^2 s}{Q^4} \left(\frac{1}{3} + 2 - 1\right) \\
&= x\left(Q_u^2 f_u(x) + (1-2Q_u)^2 f_d(x)\right) \frac{8\pi\alpha^2 s}{3Q^4}
\end{aligned}
\tag{26}
$$

For neutron the only difference is that $f_{n,u}(x) = f_d(x)$ and $f_{n,d}(x) = f_u(x)$, so we have

$$
\begin{aligned}
\frac{d^2\sigma(en \to eX)}{dx} &= x\left(Q_u^2 f_u(x) + Q_d^2 f_d(x)\right) \frac{2\pi\alpha^2 s}{Q^4} \left(\frac{1}{3} + 2 - 1\right) \\
&= x\left(Q_u^2 f_d(x) + Q_d^2 f_u(x)\right) \frac{8\pi\alpha^2 s}{3Q^4}
\end{aligned}
\tag{27}
$$

Then by definition, we have

$$
F_2^{eD}(x) = x\left(Q_u^2 + Q_d^2\right)\left(f_u(x) + f_d(x)\right) \tag{28}
$$

on the other hand from the previous result it is clear that we have

$$
F_2^{\nu D}(x) = 2x\left[f_u(x) + f_d(x)\right] \tag{29}
$$

So we have

$$
K = \frac{2}{\left(Q_u^2 + Q_d^2\right)} \tag{30}
$$

If we have $K = \frac{18}{5}$, then we know

$$Q_u^2 + Q_d^2 = \frac{5}{9}$$

This is a pretty good hint that they have fractional charges and the charges should be multiples of one third.