Color? Color!

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1 The Eightfold Way

a) Key facts about Young Tableaux:

- Vertical representations are completely antisymmetric
- Horizontal representations are completely symmetric
- To tensor product with a fundamental representation, add a box to the right or below in each possible place. A valid Young Tableau should look like upside down stairs.
- For $SU(N)$ put an N in the upper left. The number in each box should be one more than than the box to its left, and one less than the box above it.
- To find the dimension of the representation multiply all of the boxes together and divide by the hooks.

Now let's have some fun with Young Tableaux:

Please use your knowledge to prove the following important fun facts

$$\begin{aligned} 8 &= 4+2+2 \\ 27 &= 10+8+8+1 \end{aligned} \tag{1}$$

Show that they are more important for particle physicists than most of the other splittings, e.g.,

$$\begin{aligned} 8 &= 2+3+3 \\ 27 &= 20+2+2+3 \end{aligned} \tag{2}$$

Every baryon is made of 3 quarks and is a fermion, so according to Pauli the wavefunction should be completely antisymmetric under the exchange of two quarks. The wavefunction of a baryon is always factorized as the following

$$\psi_{\text{space}}\psi_{\text{spin}}\psi_{\text{flavor}}\psi_{\text{color}} \tag{3}$$

b) Let us consider $l_1 = l_2 = 0$ case (why two angular momentum?), so that ψ_{space} is always completely symmetric. In 1963, there were 9 baryons known which all had similar masses and spin $\frac{3}{2}$. According to the fun facts, which multiplet of the $SU(3)$ flavor (aka isospin) group they must live in? Which representation of the $SU(2)$ spin group?

c) Using the Young Tableau you drew, what is the symmetry of this multiplet under spin?

d) Using the Young Tableau you drew, what is the symmetry of this multiplet under flavor?

e) Now argue that there must be another symmetry, color, under which the wavefunction is completely antisymmetric.

f) If color were not there in some different universe, how many spin $\frac{3}{2}$ baryon could there be in a multiplet?

In 1964 Ω^-, the tenth baryon was discovered, and the representations of the $SU(3)$ approximate flavor symmetry or the 8-fold way were completed. We will have to do a bit more work to figure out which of the other representations are relevant. Since the wavefunction is symmetric in space, and completely antisymmetric in color, we are looking for representations that are overall symmetric in spin and flavor.

You have already argued that the singlet of $SU(3)$ color is antisymmetric. The same logic shows that the singlet of $SU(3)$ flavor is antisymmetric. The two 8's of $SU(3)$ flavor and two 2's of $SU(2)$ spin have mixed symmetry. We can choose one of the 8's to be antisymmetric under the exchange of the first two quarks and the other to be symmetric under the exchange of the first two quarks, and similarly for the the 2's.

g) Why are there no $(10, 2)$, $(8, 4)$, $(1, 4)$, or $(1, 2)$ multiplets of the flavor and spin groups $(SU(3),\ SU(2))$?

2 Magnetic Moment of the Proton and Neutron and Long Live Color

In this problem you will compute the ratio of the magnetic moment of the proton and neutron in the nonrelativistic limit. To do this, we need to construct the wavefunction of the proton and neutron. We know from the previous problem that the spin and flavor part of the wavefunction should be completely symmetric. For example, the spin and flavor parts of the wavefunction of an Ω^- with $J_3 = \frac{3}{2}$ is

$$s \uparrow s \uparrow s \uparrow \tag{4}$$

a) The symmetric combination of u, u, and d quarks is:

$$\Delta = \sqrt{\frac{1}{3}}\,[uud + udu + duu] \tag{5}$$

There are also two combinations of mixed symmetry. We can take one to be the combination antisymmetric under exchange of the first two quarks:

$$p_A = \sqrt{\frac{1}{2}}\,[ud - du]\,u \tag{6}$$

Show that the following state (which is symmetric under the exchange of the first two quarks) is orthogonal to both p_A and Δ:

$$p_S = \sqrt{\frac{1}{6}}\left[(ud + du)u - 2uud\right] \tag{7}$$

b) By making the replacements $u \rightarrow\uparrow$ and $d \rightarrow\downarrow$ we immediately get the symmetric spin-up combination of three spins:

$$\chi_S = \sqrt{\frac{1}{3}}\left[\uparrow\uparrow\downarrow + \uparrow\downarrow\uparrow + \downarrow\uparrow\uparrow\right] \tag{8}$$

as well as the spin-up combinations of mixed symmetry that are symmetric under exchange of the first two spins:

$$\chi_{MS} = \sqrt{\frac{1}{6}}\left[(\uparrow\downarrow + \downarrow\uparrow)\uparrow - 2\uparrow\uparrow\downarrow\right] \tag{9}$$

and the one antisymmetric under exchange of the first two spins:

$$\chi_{MA} = \sqrt{\frac{1}{2}}\left[\uparrow\downarrow - \downarrow\uparrow\right]\uparrow \tag{10}$$

Show that the combination

$$\sqrt{\frac{1}{2}}\left(p_S\chi_{MS} + p_A\chi_{MA}\right) \tag{11}$$

is completely symmetric. You have found the wavefunction for a spin-up proton. Ta da!

c) In the non-relativistic approximation, we can write the magnetic moment of the proton as

$$\mu_p = \sum_{i=1}^{3}\langle p\uparrow|\mu_i(\sigma_3)_i|p\uparrow\rangle \tag{12}$$

where the sum is over the constituent quarks. The (classical) magnetic moment of a point charge is

$$\mu_i = Q_i\frac{e}{2m_i} \tag{13}$$

Compute the magnetic moment of the proton in terms of the magnetic moments of the up and down quarks.

d) We can find the neutron magnetic moment by exchanging u and d. By assuming $m_u \approx m_d$, determine the ratio of the proton to neutron magnetic moments. Experimentally the ratio is measured to be $\frac{\mu_p}{\mu_n} = -1.45989806(34)$. Does your calculation agree?

e) Note that suppose again we do not have color symmetry, and we can indeed construct a totally antisymmetric proton wavefunction, such as

$$|p\uparrow\rangle = \sqrt{1}2[p_A\chi_{MS} - p_S\chi_{MA}]$$

Calculate the magnetic moment ratio again, and show that it is excluded by experiment. Long live color.

3 How the Higgs Low Energy Theorem Saves Trees: 50% accurate? We will take it.

In this problem we explore how the Higgs Low Energy Theorem makes calculations easier.

In class we saw how to choose a gauge such that the would-be Goldstone bosons are "eaten up" and we are left with one real higgs field and three of the vector bosons become massive. This choice of gauge is known as the unitary gauge and in this gauge the massive vector boson propagator is given by:

$$D_{\mu\nu\ (\mathrm{W,Z})}(q) = \frac{-i\left(\eta_{\mu\nu} - \frac{q_\mu q_\nu}{M^2_{\mathrm{W,Z}}}\right)}{q^2 - M^2_{\mathrm{W,Z}}}.$$

For large values of q^2 the propagator goes as q^0 instead of the usual q^{-2}. This scaling means that all of the divergences are worse, and loop calculations in unitary gauge are usually significantly more difficult than in other gauges. It is also hard to see that the theory is renormalizable in unitary gauge. There is another class of gauge choice, known as the R_ξ gauge, in which the physical degrees of freedom are not manifest but loop calculations are generally easier.

We start by setting:

$$H = \begin{pmatrix} w^+ \\ \frac{1}{\sqrt{2}}\left(v + h + iz\right) \end{pmatrix}.$$

where v is the Higgs VEV, h is the physical higgs, and z and $w^\pm$ are the would-be Goldstone bosons that were "eaten" in unitary gauge. Our goal is to just to draw the Feynman diagrams for $h \to \gamma\gamma$ (not compute them), so we will first determine (some of) the new vertices that are present in R_ξ gauge. We will not be interested in the Feynman rules, only which vertices are present.

a) (Skip this part at first and come back to it if you have time.) Show that the quartic higgs self-interaction term gives rise to a w^+w^-h interaction.

b) (Skip this part at first and come back to it if you have time.) Show that the higgs kinetic term gives rise to w^+w^-A, $w^\pm hW^\mp$, $w^\pm W^\mp A$, $hw^\pm W^\mp A$, and w^+w^-AA interactions.

c) Draw the diagrams that contribute to the higgs to two photon decay $h \to \gamma\gamma$ at one loop in R_ξ gauge.

d) Draw the diagrams that you would need to compute to use the Higgs Low Energy Theorem to find the higgs to two photon partial decay rate.[1] (*Hint*: Don't be afraid....) Do you think the diagrams in part (c) or part (d) are easier to compute?

[1] If you followed Sean's advice on how to become a beta person, you may have already computed some of these.